Linguistic Mapping

The Principles of Calculus I

VI

Local Linear Approximation

VI.4

Shape and Change

Classroom Exercises

Exercise 1

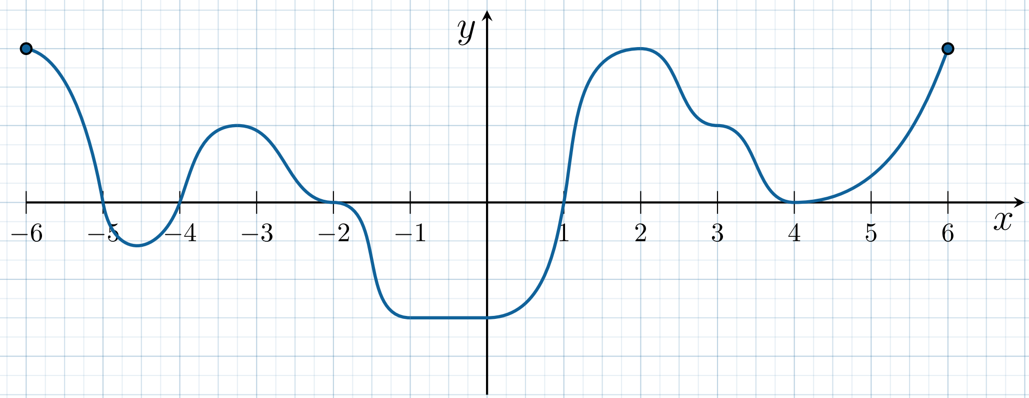
The derivative of a function on an interval determines how a function changes in a very specific way:

Theorem.

For any interval $[a, b]$ and any function f that is differentiable on (a, b) and continuous on $[a, b]$, if $f'(x)$ is positive (negative) for any x in (a, b), then f is strictly increasing (decreasing) on $[a, b]$.

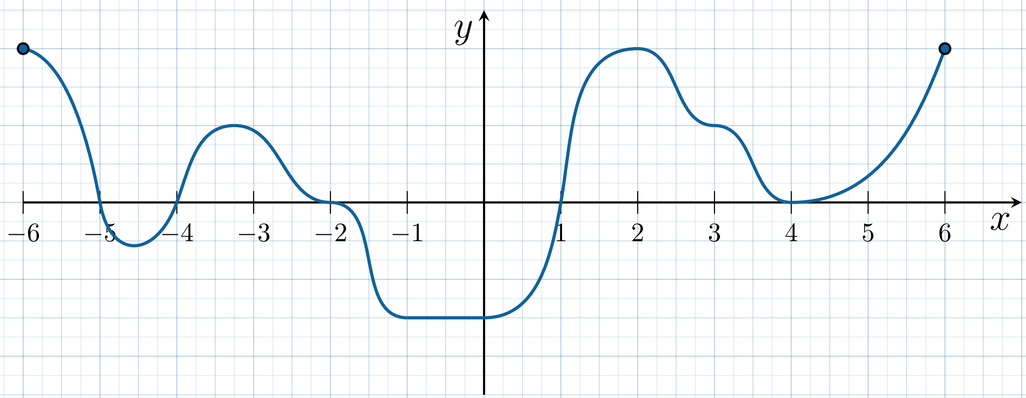
The function f is differentiable on $[-6, 6]$ and f' is sketched below:

(a) Identify all maximal intervals on which f is increasing.



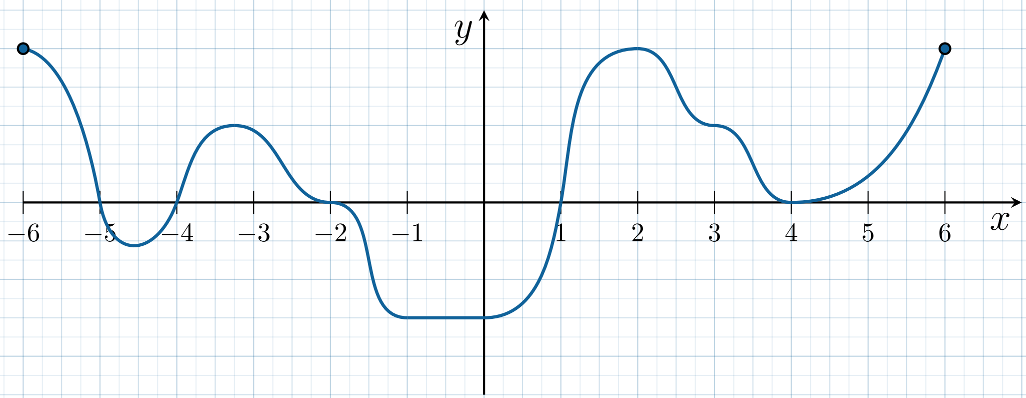
The function f is differentiable on $[-6, 6]$ and f' is sketched below:

(b) Identify all maximal intervals on which f is decreasing.



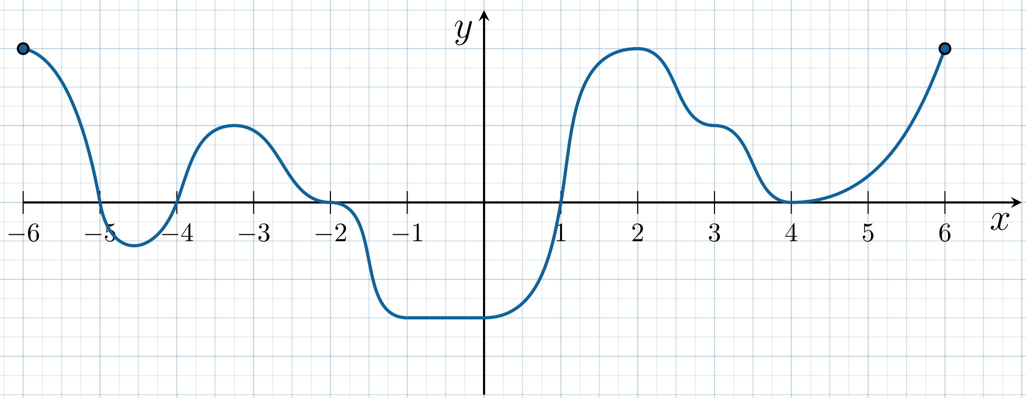
The function f is differentiable on $[-6, 6]$ and f' is sketched below:

(c) Identify all local maximum and local minimum.



The function f is differentiable on $[-6, 6]$ and f' is sketched below:

(d) Describe how the function f may look at each x_0 for which f' is zero, but where x_0 is neither a local minimum or local maximum.



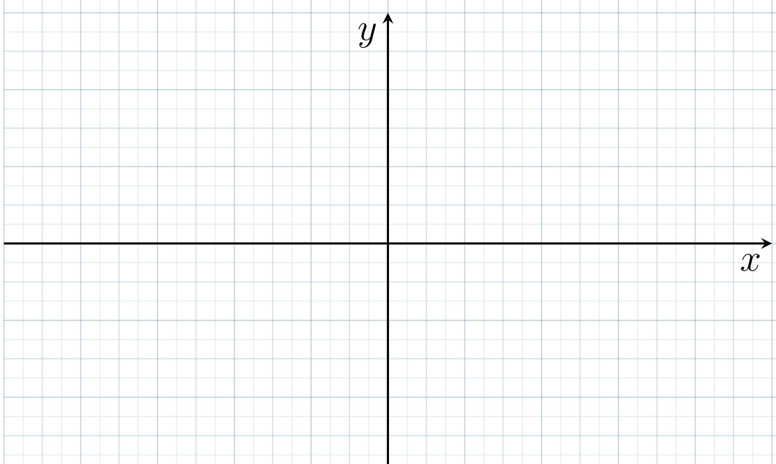
Exercise 2

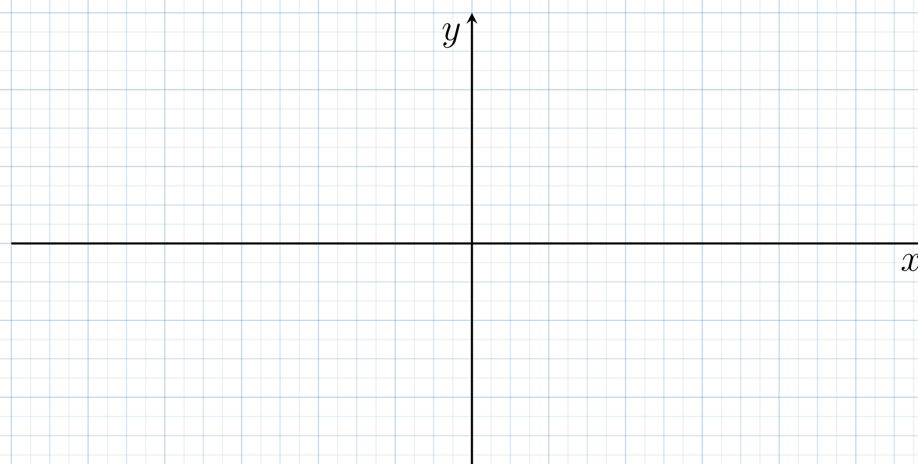
A sketch of a function's derivative is useful for identifying certain features of the original function. For each of these choices of function f with derivative f' specified, identify all points in $\mathbb{R}$ where f has a local maximum or minimum and determine all maximal intervals for which f is increasing and decreasing:

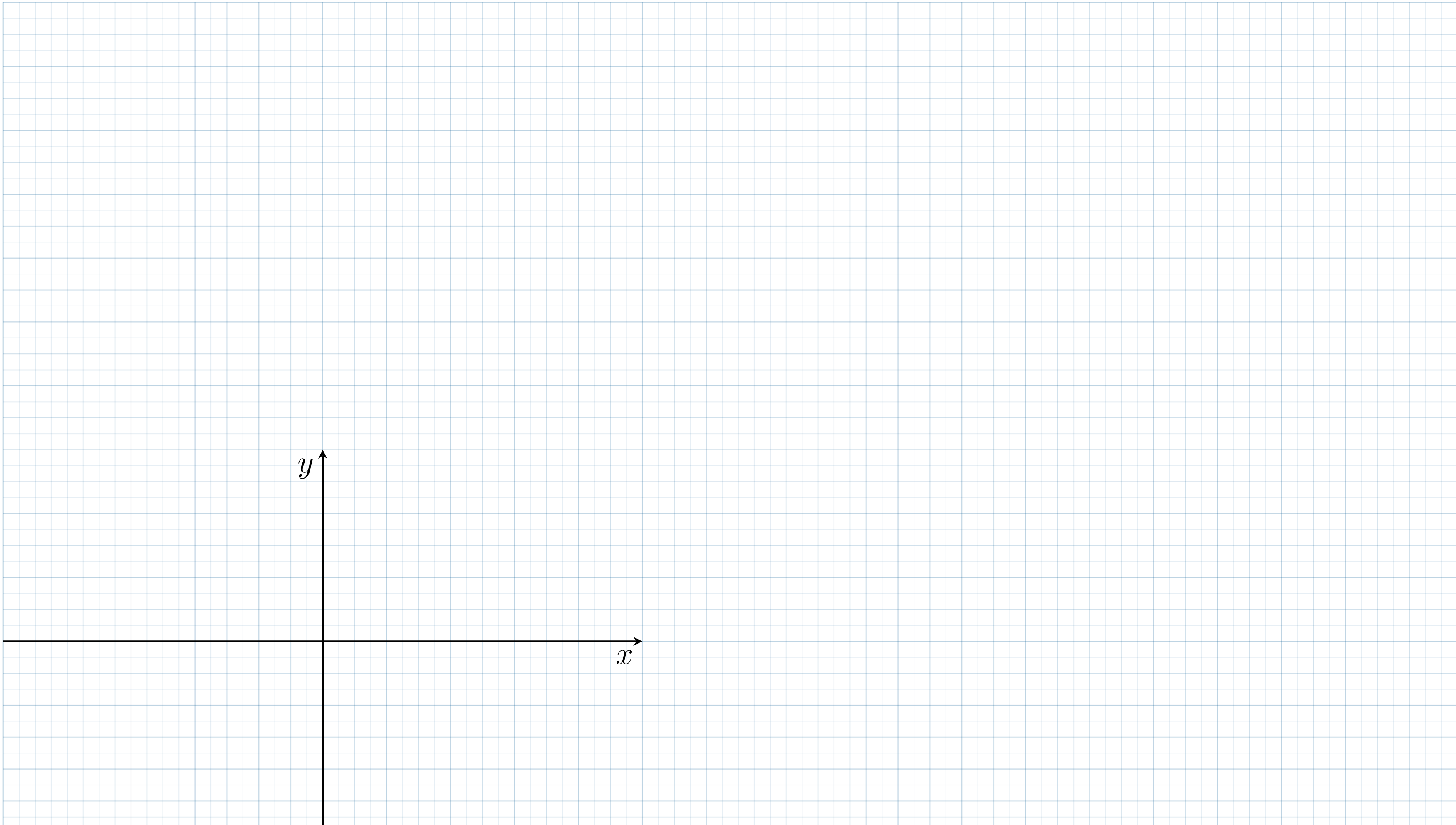
(a) $f'(x) = -4(x+3)^2(x+1)^4(x-1)^3(x-4)^4$;

(b) $f'(x) = \mathrm{e}^x + 4$;

(c) $f'(x) = \frac{x+3}{(x-2)^2}$;



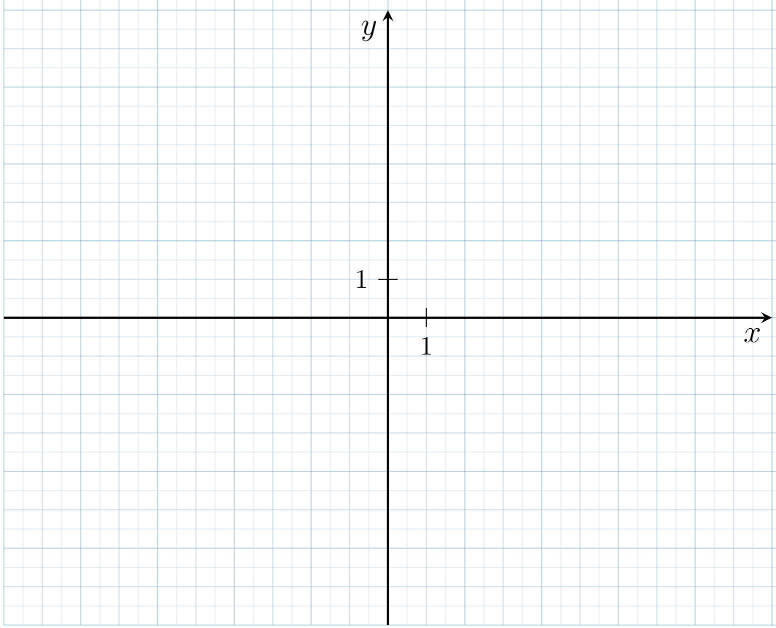



y
x

Take f to be the function that is given by

$$f(x) = |x|(x^2 - 6x - 7). \qquad \left(\textbf{Note:}\quad x^2 - 6x + 7 = (x+1)(x-7).\right)$$

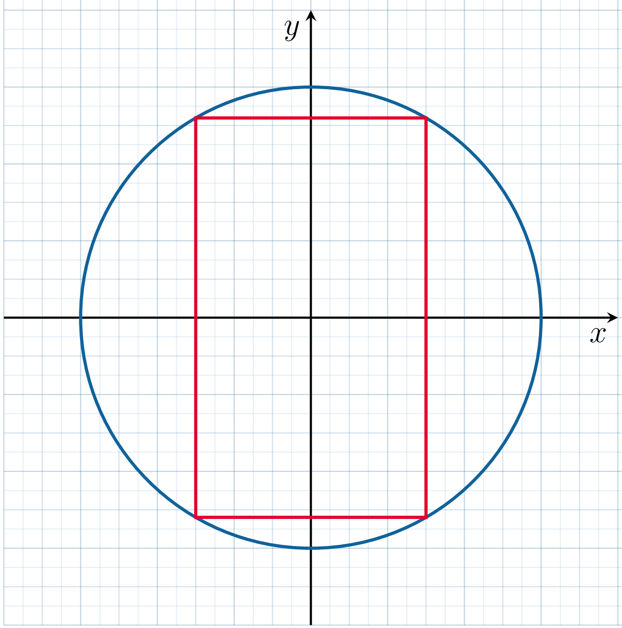
Determine the derivative of f to identify all maximal intervals on which f is increasing and decreasing.



Exercise 4

In this exercise, we will determine the dimensions of the cylinder that has the largest possible volume and that fits into a sphere of radius 18 inches.

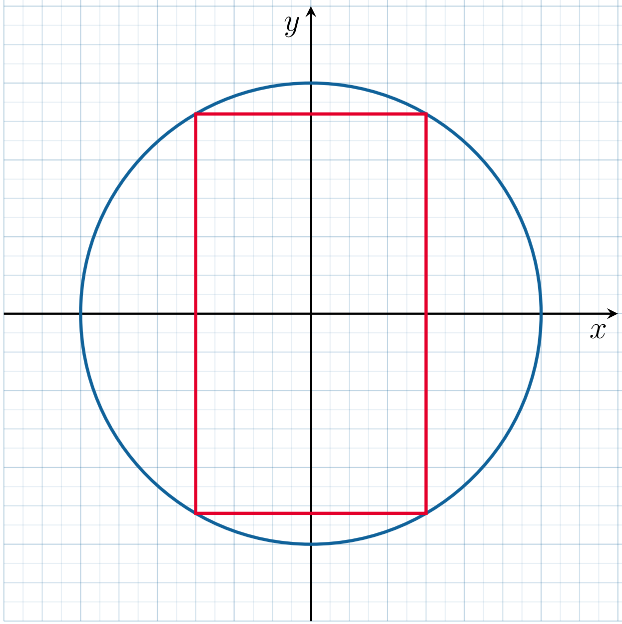
(a) Take r to be the radius of the cylinder. Determine a function in r that gives the volume of a cylinder that fits in a sphere of radius 18 inches.



(b) Determine the derivative of the function in (a).

(c) Determine the feature in the derivative we are interested in so that the largest possible volume is obtained.

(d) Determine the radius that yields the largest volume.



Exercise 5

A particle moves along an ellipse at a constant angular speed and its path of motion is given by c, where

$$c(t) = (3\cos(4t), 5\sin(4t)).$$

(a) Explain what it means to say that the angular speed is constant.

(b) At what time points is the speed of the particle maximal?

(c) At what time points is the speed of the particle minimal?

Exercise 6

The derivative of a function is itself a function. It is natural to ask if the derivative has a derivative and if so, identify the information this *second derivative* captures.

(a) State what it means for a function to be *twice differentiable* at x_0.

(b) Write out the notation for the derivative of the derivative of a function in two different ways, using “prime” notation and Leibniz notation.

(c) State in plain English the meaning of first order and second order information about a function f.

(d) State what information the second derivative gives about the derivative of f.

(e) State what information the second derivative gives about the original function given that the original function describes the position of a particle on a line.

Exercise 7

Take f and g to be the functions given by

$$f(t) = \tfrac{1}{4}t^4 - 2t^3 + 4t^2 + 1 \quad \textbf{and} \quad g(t) = \mathrm{e}^{-t^2}.$$

Interpret each of these function as giving the position of a particle at time t that is moving on a horizontal line.

(a) Identify all maximal intervals on which the particle is moving to the right or moving to the left.

(b) Identify all maximal intervals on which the particle is accelerating to the right or accelerating to the left.

(c) Simulate the motion of the particle.

Bounds on the first derivative of a function on an interval guarantee that f cannot change by more than a certain amount in a certain interval. Similarly, bounds on the second derivative guarantee that the first derivative cannot change by more than a certain amount on that same interval. These bounds lead to estimates for f. Complete the statement of the second order control theorem.

Theorem (Second Order Control Theorem).

For any interval I, any real number M and m, and any function f that is twice differentiable on I, if for each z in I,

$$f''(z) \leq M \quad \text{and} \quad f''(z) \geq m,$$

then for any x in I,

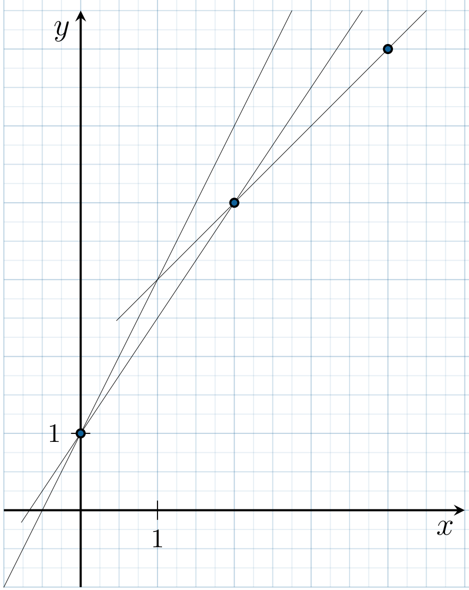
$$f(x) \leq \boxed{} + \boxed{}(x-a) + \boxed{}(x-a)^2,$$

$$\text{and} \quad f(x) \geq \boxed{} + \boxed{}(x-a) + \boxed{}(x-a)^2.$$

Take f to be a twice differentiable function on $[0,4]$ and suppose that $|f''(x)|$ is bounded above by K for all x in $[0,4]$. Roughly sketch the smallest region in the plane that is guaranteed to contain f, where

$$K=1, \quad f(0)=1, \quad f'(0)=2, \quad f(2)=4, \quad f'(2)=\tfrac{3}{2}, \quad f(4)=6, \quad \text{and} \quad f'(4)=1.$$

Roughly sketch means to not calculate the intersections of various parabolas, see this: (simulation).



Exercise 10

The second derivative of a function determines a function's shape. For this exercise, take f and g to be the functions given by

$$f(x) = x^2 \quad \text{and} \quad g(x) = \sqrt{x}.$$

(a) Sketch f and g.

(b) State whether the functions f and g "bend" in the same way. Do they "curve upwards" or "curve downwards"?

(c) Sketch secant lines for f and g and determine whether each secant line lies above or below its respective function.

Exercise 11

The second derivative of a function determines a function's shape. It is helpful to develop some language to express the idea.

(a) Explain what it means for a function to be *convex* on an interval I.

(b) Explain what it means for a function to be *concave* on an interval I.

(c) Sketch an example of a function that is convex and an example of a function that is concave.

(d) Complete the following statements about a function f:

- If f'' is $\boxed{}$ on an interval I, then f is convex on I.
- If f'' is $\boxed{}$ on an interval I, then f is concave on I.
- If p is a point in the domain of a function f at which the function's concavity changes, then it is an $\boxed{}$

(e) Sketch an example of a function that has an inflection point.

(f) State alternative vocabulary for the words convex and concave that may be seen in some calculus textbooks.

For each of these choices of function f, identify all inflection points of f and give all maximal intervals on which f is convex or concave:

(a) $f(x) = x^3 - 7x^2 + 16x - 12$;

(b) $f(x) = |x|(x^2 - 6x - 7)$.

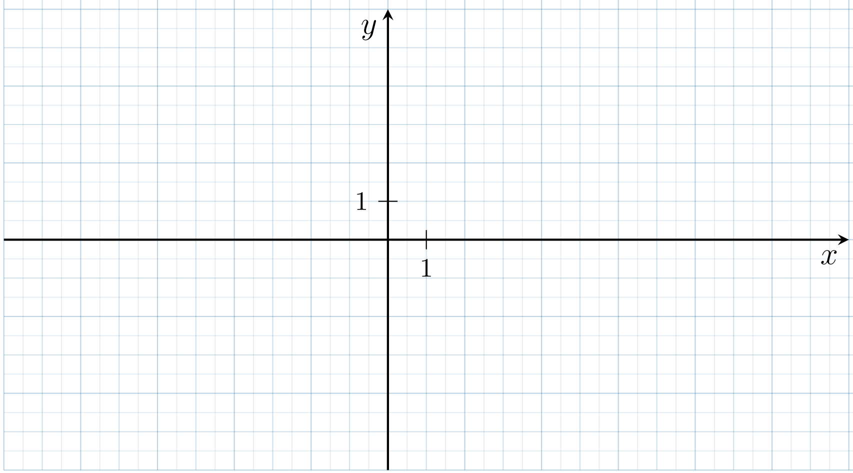
Use technology to sketch these functions and compare the sketch to your answers.

Exercise 13

Incorporate in stages the zeroth, first, and second order information to sketch a function f with the following properties.

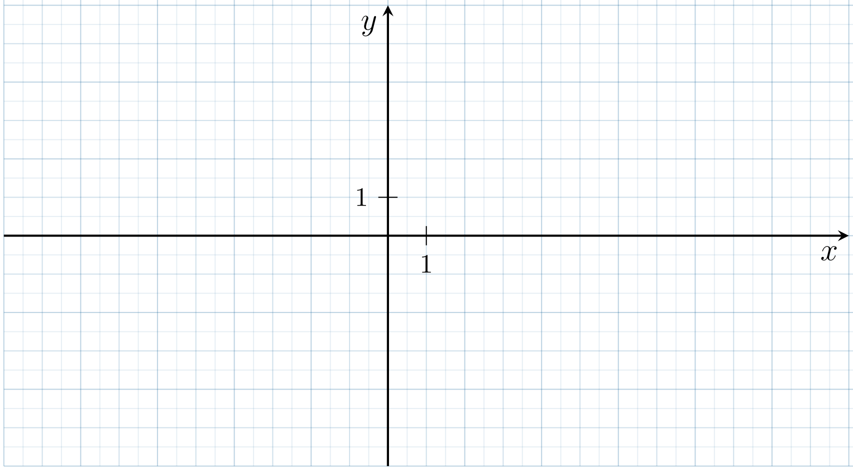
(a) f is continuous on $(-\infty, 3) \cup (3, \infty)$ which is its domain and the following hold:

- the zero set of f is the set $\{-6, -4, 4, 8\}$,
- the vertical line that intersects $(3, 0)$ is a vertical asymptote for f,
- f is asymptotically equal to pow_2,
- $f(-5) = -1$, $f(-2) = 3$, $f(-1) = 2$, $f(1) = 4$, and $f(6) = 4$.



(b) The function f additionally has this first order information:

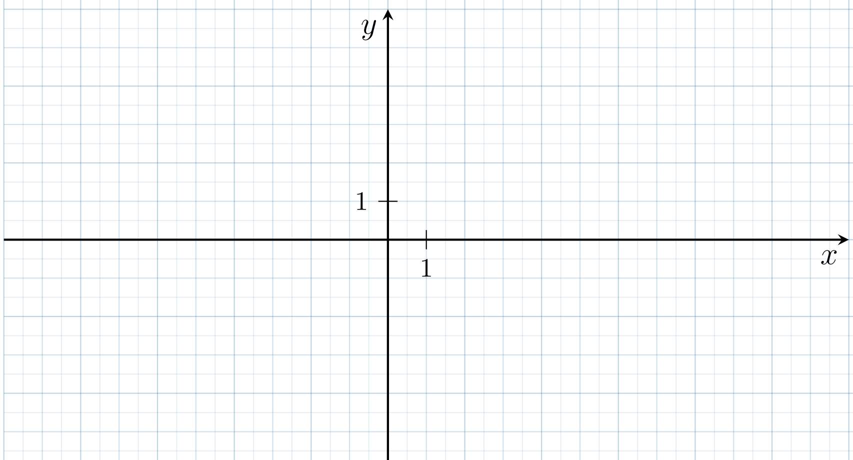
- f' is defined on the set $(-\infty, -1) \cup (-1, 3) \cup (3, 8) \cup (8, \infty)$,
- the zero set of f' is the set $\{-5, -2, 1, 6\}$,
- f' is positive on the set $(-5, -2) \cup (-1, 1) \cup (1, 3) \cup (3, 6) \cup (8, \infty)$,
- f' is negative on the set $(-\infty, -5) \cup (-2, -1) \cup (6, 8)$.



(c) The function f additionally has this second order information:

- f'' is defined on the set $(-\infty, -1) \cup (-1, 3) \cup (3, 8) \cup (8, \infty)$,
- the zero set of f'' is the set $\{-3, 1\}$,
- f'' is positive on the set $(-\infty, -3) \cup (1, 3) \cup (8, \infty)$,
- f'' is negative on the set $(-3, -1) \cup (-1, 1) \cup (3, 8)$.

Label any local maxima, minimum, and any inflection points.

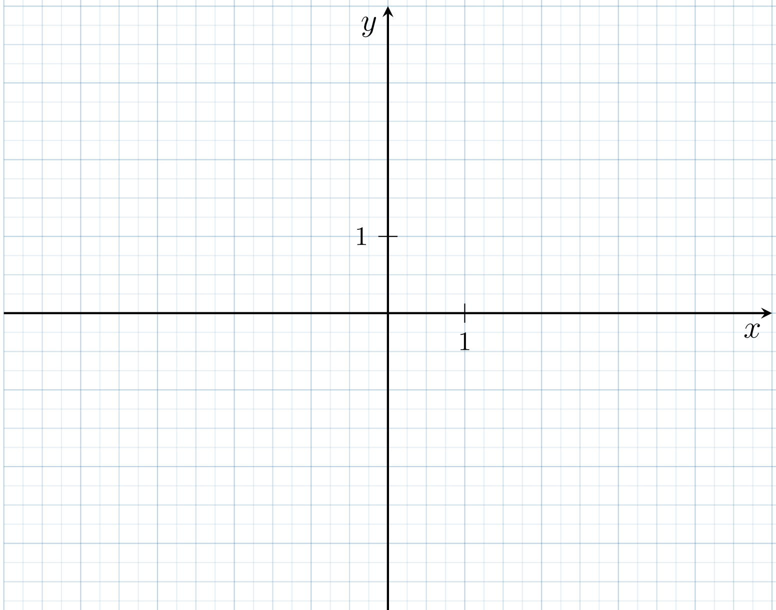


Exercise 14

Combining first order and second order information facilitates sketching functions. For each of these choices of function f, sketch f and label any local maxima, minima, and inflection points:

(a) $f(x) = 2x^3 - 3x^2 - 12x - 8$. **Note:** $f'(x) = 6(x+1)(x-2)$ and $f''(x) = 6(2x-1)$.

How does the first order information assist you in determining the roots of f when using a method such as the bisection method?



(b) $f(x) = |x|\mathrm{e}^{-x}$

Note: $$f'(x) = \begin{cases} (x-1)\mathrm{e}^{-x} & \textbf{if } x < 0 \\ (1-x)\mathrm{e}^{-x} & \textbf{if } x > 0, \end{cases}$$

$$f''(x) = \begin{cases} (2-x)\mathrm{e}^{-x} & \textbf{if } x < 0 \\ (x-2)\mathrm{e}^{-x} & \textbf{if } x > 0. \end{cases}$$