Linguistic Mapping

The Principles of Calculus I

VI

Local Linear Approximation

VI.3

Rigidity and the Local Linear Approximation

Classroom Exercises

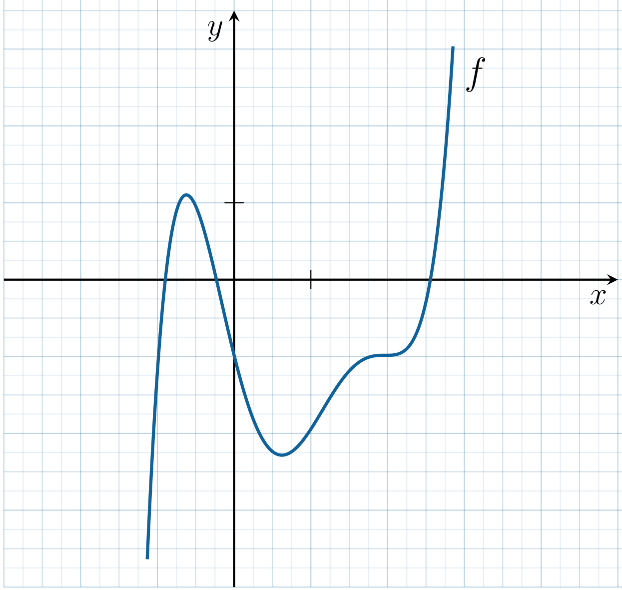
Exercise 1

Analyzing the derivative of a function can reveal certain features of the original function. Take f to be a differentiable function whose sketch is given below.

(a) Identify any local maxima and local minima of f.

(b) Sketch the lines tangent to f at the local maxima and local minima. Identify the slope of these lines.

(c) Sketch a tangent line to f at $(2, f(2))$ and identify its slope.



Exercise 2

Fermat's theorem is useful for identifying the extremal values for differentiable functions.

(a) State Fermat's theorem.

(b) Identify how Fermat's theorem applies to the previous exercise.

(c) Take f to be the function given by

$$f(x) = |x - 1|.$$

Determine if Fermat's theorem can be applied to f.

(d) Determine the extremal values of f.

Exercise 3

Take f to be a continuous function on the interval $[a, b]$ such that it is differentiable at all points (a, b) except possibly on a finite subset S.

(a) Explain why f is guaranteed to attain a maximum or minimum in $[a, b]$.

(b) Explain in plain English what the zero set of f' is and why it is useful to identify the local extrema of f.

(c) Explain why the set S is important to identify.

(d) Explain in plain English what the critical points of f are and why identification of this set is important.

Exercise 4

For each of these choices of interval $[a, b]$ and function f, determine all points at which f attains its maximum and minimum values on $[a, b]$:

(a) $f(x) = (x + 3)(x - 2)(x + 4)$ and $[a, b] = [-5, 3]$;
(b) $f(x) = |x(x + 2)^2|$ and $[a, b] = [-3, 2]$.

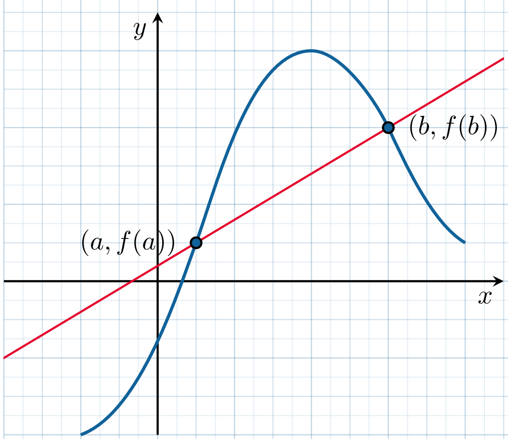
Exercise 5

Fermat's theorem and the extreme value theorem for continuous functions on a closed and bounded interval lead to important results on the rigidity of differentiable functions.

(a) State Rolle's Theorem.

(b) State the Mean Value Theorem.

(c) Use the provided picture to assist in giving a proof of the Mean Value Theorem that uses Rolle's Theorem.



Exercise 6

The Mean Value Theorem has many important consequences. One consequence is a theorem that bounds the difference between function values on an interval, the length of the interval, and a bound on the derivative of the function.

(a) State the Bounded Derivative Estimate.

(b) Explain in plain English the meaning of the Theorem.

(c) Sketch a proof of the theorem.

(d) Conclude that any function that is differentiable on an interval I and has a bounded derivative has a modulus of continuity function.

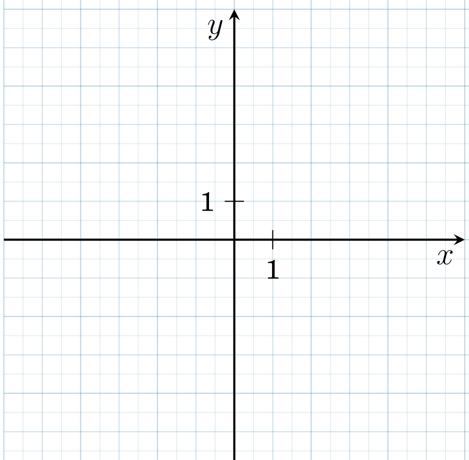
Exercise 7

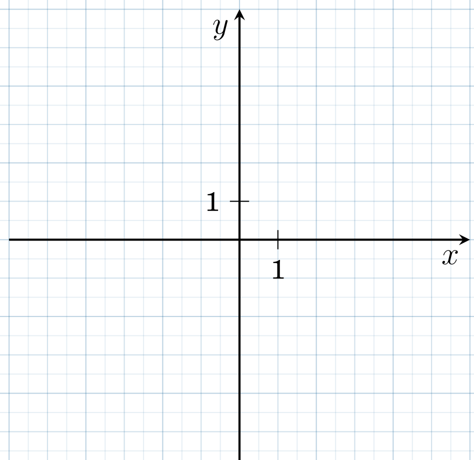
Take f to be a function that is differentiable on $[-4, 5]$ so that for all x in $[-4, 5]$,

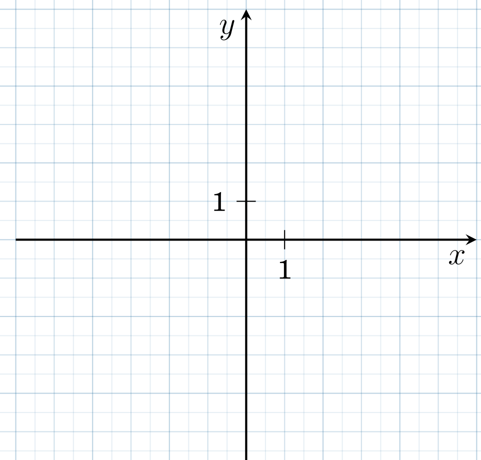
$$|f'(x)| \leq \frac{1}{3}.$$

For each of these choices of specified points in f, determine the smallest range of values that is guaranteed to contain $f([-4, 5])$ and sketch the smallest region that is guaranteed to contain f.

(a) $f(-4) = 1$;   **(b)** $f(5) = 2$;   **(c)** $f(2) = 2$;

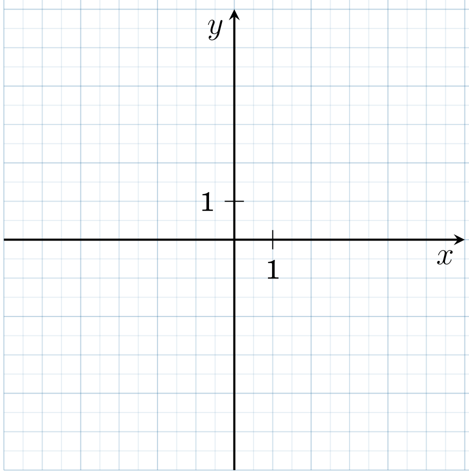


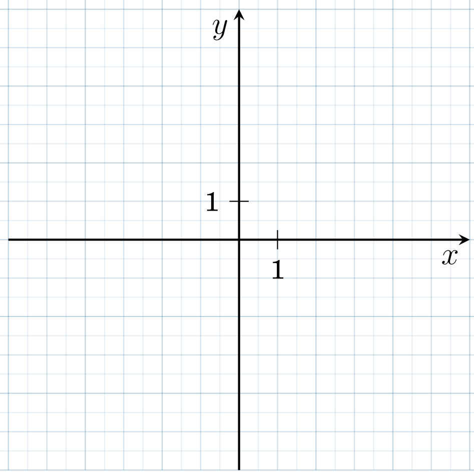




(d) $f(-4) = 1$ **and** $f(5) = 2$;

(e) $f(-4) = 1$, $f(2) = 2$, **and** $f(5) = 2$.





Exercise 8

The mean value theorem implies this theorem about the rigidity of differentiable functions:

> Any function whose derivative is equal to the zero function is entirely determined by its value at a single point.

(a) Formally state this theorem (uniqueness of functions with everywhere zero derivative).

The mean value theorem implies this theorem about the rigidity of differentiable functions:

> Any function whose derivative is equal to the zero function is entirely determined by its value at a single point.

(b) Sketch a proof of this theorem.

The mean value theorem implies this theorem about the rigidity of differentiable functions:

> Any function whose derivative is equal to the zero function is entirely determined by its value at a single point.

(c) Take I to be an interval and take f and g to be two differentiable functions on I such that for any x in I,

$$f'(x) = g'(x).$$

Use this relationship to establish a relationship between f and g.

Exercise 9

Differentiation is an operation that takes a function f to its derivative f'. In this exercise we define an operation that takes the function f to another function F so that the derivative of F is f.

(a) Define what it means for a function F to be an *antiderivative* of f.

(b) If a function has an antiderivative, is the antiderivative unique?

(c) Identify the meaning of the *indefinite integral* of f, usually denoted by $\int f(x)\,\mathrm{d}x$, and explain the relationship between the indefinite integral of f and an antiderivative of f.

(d) State why someone might refer to antidifferentiation as *reverse differentiation*.

Exercise 10

The various formulas for differentiation imply certain helpful formulas for antidiferentiation. Take A to be a constant and r to be a real number not equal to 0 or -1. Determine the following:

- $\int A \, \mathrm{d}x =$ [ ]
- $\int x \, \mathrm{d}x =$ [ ]
- $\int x^r \, \mathrm{d}x =$ [ ]
- $\int \sin(x) \, \mathrm{d}x =$ [ ]
- $\int \cos(x) \, \mathrm{d}x =$ [ ]
- $\int \sec^2(x) \, \mathrm{d}x =$ [ ]

The various formulas for differentiation imply certain helpful formulas for antidiferentiation. Take A to be a constant and r to be a real number not equal to 0 or -1. Determine the following:

- $\int \sec(x)\tan(x)\,\mathrm{d}x =$ ______
- $\int \frac{1}{1+x^2}\,\mathrm{d}x =$ ______
- $\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x =$ ______
- $\int -\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x =$ ______
- $\int \mathrm{e}^x\,\mathrm{d}x =$ ______
- $\int \frac{1}{x}\,\mathrm{d}x =$ ______

Exercise 11

For each of these choices of function f, identify an antiderivative for f and determine $\int f(x)\,\mathrm{d}x$:

(a) $f(x) = 10x^2 + \frac{20}{x} + \sqrt{x}$;

(b) $f(x) = 5\sin(x) + \sec(x)\tan(x) + \frac{10}{1+x^2} - 10$;

(c) $f(x) = \mathrm{e}^x + \cos(x) + 4\sec^2(x) + \arctan(4)$.

Exercise 12

The formulas for differentiation imply helpful formulas for antidiferentiation. Take F and G to be antiderivatives on an interval I for the functions f and g, respectively.

(a) Determine $(FG)'$.

(b) Use the mean value theorem together with (a) to derive the following equality:

$$\int F(x)g(x)\,\mathrm{d}x = F(x)G(x) - \int f(x)G(x)\,\mathrm{d}x.$$

(c) Explain why someone may refer to this formula as the *reverse product rule*.

Exercise 13

Take f to be the function given by

$$f(x) = x\cos(x).$$

Identify an antiderivative for f and determine $\int f(x)\,\mathrm{d}x$.

The formulas for differentiation imply helpful formulas for antidiferentiation. Take F and G to be antiderivatives for the functions f and g, respectively, on some appropriate intervals.

(a) Determine $(F \circ G)'$.

(b) Use the mean value theorem together with (a) to derive the following equality:

$$\int f(G(x))g(x)\,\mathrm{d}x = F(G(x)) + C.$$

(c) Explain why someone may refer to this formula as the *reverse chain rule*.

Exercise 15

For each of these choices of function f, identify an antiderivative for f and determine $\int f(x)\,\mathrm{d}x$:

(a) $f(x) = 6x\sin(3x^2+1)$;

(b) $f(x) = 7x\sec^2(5x^2)$;

(c) $f(x) = \frac{40x+10}{\sqrt{10x^2+5x+1}}$.

Exercise 16

In many problems that arise in physical applications, information about the velocity of a path may be known, but little information may be known directly about the path itself. Up to some initial location in the plane, antidifferentiation permits a reconstruction of a path from knowledge of the velocity associated to the path. Take $v(t)$, the velocity a path c at time t, to be given by

$$v(t) = \langle 5t + 2, -4 \rangle.$$

(a) Write $c(t)$ and $v(t)$ abstractly with respect to component functions.

(b) Determine all possible component functions for c.

(c) Given that

$$c(1) = (2, 5),$$

reconstruct the path c and simulate the motion of the particle together with the particle's velocity vector.

Local linear approximation can facilitate the evaluation of limits that cannot immediately be determined by direct application of the limit laws. For this exercise, take I to be an interval that contains x_0 and f and g to be functions defined on $I \setminus \{x_0\}$. Each of these limits is said to have *indeterminate form*:

(a) $\lim\limits_{x\to x_0} \frac{f(x)}{g(x)}$ where either

- $\lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} g(x) = \square$ or $\lim\limits_{x\to x_0} f(x) = \square$ and $\lim\limits_{x\to x_0} g(x) = \square$;

(b) $\lim\limits_{x\to x_0} (f(x) - g(x))$ where either

- $\lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} g(x) = \square$ or $\lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} g(x) = \square$;

(c) $\lim\limits_{x\to x_0} (f(x))^{g(x)}$ where either

- $\lim\limits_{x\to x_0} f(x) = \square$ and $\lim\limits_{x\to x_0} g(x) = \square$, or $\lim\limits_{x\to x_0} f(x) = \square$ and $\lim\limits_{x\to x_0} g(x) = \square$

  or $\lim\limits_{x\to x_0} f(x) = \square$, f is non-negative, and $\lim\limits_{x\to x_0} g(x) = \square$.

Note that these definitions extend to limits at infinity and directional limits.

Check the box next to each of these limits that has an indeterminate form:

(a) $\lim_{x\to 0} \frac{\cos(x^2)}{x}$; ☐

(b) $\lim_{x\to 1} \frac{\sin(\pi x)}{x-1}$; ☐

(c) $\lim_{x\to\infty} \left(\sqrt{x+4} - \sqrt{x+3}\right)$; ☐

(d) $\lim_{x\to 5} \frac{x-5}{(x+5)^2}$; ☐

(e) $\lim_{x\to \frac{3}{2}^+} \frac{\ln(x-\frac{3}{2})}{\tan(\pi x)}$; ☐

(f) $\lim_{x\to 0} x^3 \ln(x^4)$; ☐

(g) $\lim_{x\to 0^+} x^x$; ☐

(i) $\lim_{x\to\infty} \frac{x^2}{e^{-5x}}$; ☐

Exercise 19

Here is a version of L'Hopital's rule. Complete the information:

For any interval (a, b), any x_0 in (a, b), and any functions f and g that are continuous on $[a, x_0) \cup (x_0, b]$, differentiable on $(a, x_0) \cup (x_0, b)$, and either

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = \boxed{},$$

or

$$\lim_{x \to x_0} f(x) = \boxed{} \quad \text{and} \quad \lim_{x \to x_0} g(x) = \boxed{},$$

existence of the limit

$$\lim_{x \to x_0} \boxed{}$$

implies the existence of

$$\lim_{x \to x_0} \frac{f(x)}{g(x)},$$

and, this case, both limits are equal.

Exercise 20

Verify that these limits have indeterminate forms and use L'Hopital's rule to evaluate these limits:

(a) $\lim_{x \to 5} \frac{x^3 - 23x - 10}{x - 5}$;

(b) $\lim_{x \to \infty} \frac{x^4}{e^x}$;

(c) $\lim_{x \to \infty} \frac{\ln(x)}{x}$;

(d) $\lim_{x \to \infty} \left((8x^3 + 3x^2 + 10x + 1)^{\frac{1}{3}} - 2x \right)$.

In the case of (b) and (c), what can be said about the relative growth of the functions?

Exercise 21

The root test is an important test for determining the convergence of series.

(a) Recall and write out the statement of the root test.

(b) Use L'Hopital's rule to determine

$$\lim_{n\to\infty} n^{\frac{1}{n}}.$$

The root test is an important test for determining the convergence of series.

(c) For any natural number k, use L'Hopital's rule to determine

$$\lim_{n\to\infty} (n^k)^{\frac{1}{n}}.$$

The root test is an important test for determining the convergence of series.

(d) Use both the root and ratio tests to determine the convergence of the series $\sum \frac{n^k}{2^n}$.

The root test is an important test for determining the convergence of series.

(e) Use both the root and ratio tests to determine the convergence of the series $\sum \frac{2^n}{n^n}$.