Linguistic Mapping

The Principles of Calculus I

VI

Local Linear Approximation

VI.2

Differentiating Elementary Functions

Classroom Exercises

Exercise 1

Reflection across pow_1 preserves tangential intersections in the setting of functions that are quotients of linear polynomial functions. As an example, take f to be the function that is given by

$$f(x) = \frac{2x-1}{x-3}.$$

(a) Using only algebraic methods, identify the line L that is tangent to f at $(4,7)$.

(b) Determine an equation for L^{-1}.

Reflection across pow_1 preserves tangential intersections in the setting of functions that are quotients of linear polynomial functions. As an example, take f to be the function that is given by

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(c) Determine f^{-1} and directly determine the line that is tangent to f^{-1} at $(7,4)$.

Reflection across pow_1 preserves tangential intersections in the setting of functions that are quotients of linear polynomial functions. As an example, take f to be the function that is given by

$$f(x) = \frac{2x-1}{x-3}.$$

(d) Use limits to calculate $f'(4)$ and $(f^{-1})'(7)$.

Exercise 2

Theorem (Inverse function theorem).

For any open interval I and any continuous invertible function f with domain I, if x_0 is in I, f is differentiable at x_0, and $f'(x_0)$ is not equal to 0, then f^{-1} is differentiable at $f(x_0)$ and

$$\left(f^{-1}\right)'(f(x_0)) = \frac{1}{f'(x_0)}.$$

(a) For any $a \in (0,1) \cup (1,\infty)$, identify the domain of each of these functions and its corresponding inverse: $\log_a$, $\arccos$, $\arcsin$, and $\arctan$.

(b) Identify the derivatives of the functions $\exp_a$, $\cos$, $\sin$, and $\tan$.

(c) Use the inverse function theorem to identify the derivatives of the functions $\log_a$, $\arccos$, $\arcsin$ and $\arctan$.

Exercise 3

For each of these choices of function f, carefully decompose f into simpler functions, write down the derivatives of these simpler functions, and then use the rules for differentiation to determine a formula for $f'(x)$:

(a) $f(x) = \arctan(2x^2 + 10)\ln(5x + 2)$;

(b) $f(x) = \frac{3x+1}{\arcsin(x)+x^2+5}$.

Exercise 4

Careful study of the logarithm function and its algebraic properties reveals alternative methods for differentiating functions.

(a) Denote the absolute value function by

$$\mathrm{abs}(x) = |x|.$$

Determine the derivative of abs and its domain.

(b) Determine the derivative of $\ln \circ \text{abs}$.

(c) Take x_0 to be a real number and f to be a function that is differentiable at x_0. Suppose further that $f(x_0)$ is nonzero. Determine the derivative at x_0 of the function g that is given by

$$g = \ln \circ \operatorname{abs} \circ f.$$

(d) Show that

$$g'(x_0) = \frac{f'(x_0)}{f(x_0)}, \quad \text{and so} \quad f'(x_0) = g'(x_0)f(x_0).$$

Some textbooks refer to differentiation performed in this way as *logarithmic differentiation*.

Exercise 5

Take f and g to be the rational functions given by

$$f(x) = (x+4)(x-3)(x^2+6)(x^4+x^2+10) \quad \text{and} \quad g(x) = \frac{(x^2+6)(x-1)(5x+4)^2}{(x+5)(x^5+2x^2-1)}.$$

Use logarithmic differentiation to determine $f'(x)$ and $g'(x)$.

(a) Describe in words the process for differentiating f and g without using logarithmic differentiation. For example, identify and quantify the derivative rules needed to differentiation f and g.

(b) Some would suggest it is “easier” to use logarithmic differentiation for these choices of f and g. Identify a way to quantify this opinion by analyzing the differences in (a) and (b).

Exercise 6

Recall that for any positive function f and any function g,

$$f(x)^{g(x)} = \mathrm{e}^{g(x)\ln(f(x))}.$$

Use this equality to determine $a'(x)$ and $b'(x)$, where

(a) $a(x) = x^{x^2}$;

(b) $b(x) = (1 + \sin^2(x))^{4x+1}$.

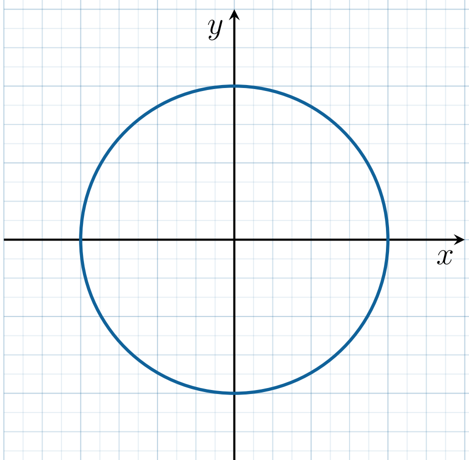
Exercise 7

Covarying quantities are not necessarily functionally related. The unit circle $\mathscr{C}$, for example, is the set of all points that satisfy

$$x^2 + y^2 = 1.$$

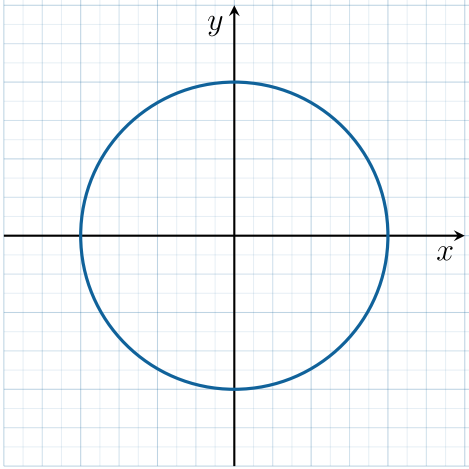
(a) Explain why the set of points is not a function.

(b) Take p to be in $\mathscr{C} \setminus \{(1,0),(-1,0)\}$. Draw an open rectangle R that contains p so that no two distinct points in $\mathscr{C} \cap R$ have the same x-coordinate.



(c) Explain why the y-coordinates of points in $\mathscr{C} \cap R$ are functions of the x-coordinates.

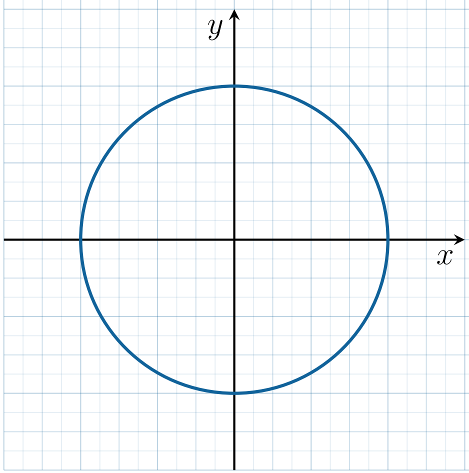
(d) Write y explicitly as a function of x.



(e) Take p to be in $\mathscr{C} \setminus \{(0,1),(0,-1)\}$. Draw an open rectangle R that contains p so that no two distinct points in $\mathscr{C} \cap R$ have the same y-coordinate.

(f) Explain why the x-coordinates of points in $\mathscr{C} \cap R$ are functions of the y-coordinates.

(g) Write x explicitly as a function of y.



Exercise 8

Suppose that an equation involving points in the plane has a solution set S. Suppose further that there is a p_0 in S and open rectangle R with these properties:

- The set $S \cap R$ contains p_0;
- There is a continuous path c that parameterizes $S \cap R$;
- There is an open interval I in $\mathcal{D}(c)$ and a t_0 in $\mathcal{D}(c)$ so that $c(t_0)$ is equal to p_0;
- The path c is differentiable at t_0 and $c'(t_0)$ is not the zero vector.

In this case, the vector $c'(t_0)$ defines a line L that is tangent to $S \cap R$ at p_0. In fact, this defines the idea of tangency to $S \cap R$ at p_0.

For this exercise, specialize to the equation

$$\sin\left(\pi\left(x+y\right)\right) - xy = 0.$$

Assume that there is a path c that is differentiable at the point $(1, 0)$ and that parameterizes the solution set of the equation.

For this exercise, specialize to the equation

$$\sin\left(\pi\left(x+y\right)\right) - xy = 0.$$

Assume that there is a path c that is differentiable at the point $(1, 0)$ and that parameterizes the solution set of the equation.

(a) Use the equality

$$c(t) = (x(t), y(t))$$

to rewrite the above equation using the local linear approximations of $x(t)$ and $y(t)$ at t_0.

(b) Compute the ratio $\frac{y'(t_0)}{x'(t_0)}$ up to $\mathrm{o}(t - t_0)$ terms.

(c) Take limits as t tends to t_0 to determine the slope of the line tangent to the solution set at $(1, 0)$.

(d) Identify an equation for the line L that is tangent to $S \cap R$ at $(1, 0)$ and use a computer to sketch the solution set as well as L.

Exercise 9

Alternative notation for a concept can provide additional clarity or insight.

(a) Explain what $\dfrac{\mathrm{d}f}{\mathrm{d}x}$ means. Note: This is the Leibniz notation.

Alternative notation for a concept can provide additional clarity or insight.

(b) Explain what $\left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_{x_0} f(x)$ and $\frac{\mathrm{d}f}{\mathrm{d}x}(x_0)$ mean.

(c) Explain what $\frac{\mathrm{d}}{\mathrm{d}x}$ and $\left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_{x_0}$ mean.

(d) Compute

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 + 5x^3 - \sin(x)\cos(x)\right).$$

How would you write this out without Leibniz notation?

(e) What is the advantage of using Leibniz notation?

(f) How might Leibniz notation lead to misunderstandings?

Exercise 10

It is helpful to rewrite the decomposition rules for differentiation using Leibniz notation.

(a) Write out the linearity of differentiation using Leibniz notation.

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(b) Write out the product rule using Leibniz notation.

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(c) Write out the chain rule using Leibniz notation.

It is helpful to rewrite the decomposition rules for differentiation using Leibniz notation.

(d) Evaluate $\left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_{x=0} \sqrt{\exp(4x) + \sin(4x^2)}$.

Exercise 11

Not every equation that relates two variables can be explicitly rewritten in the following form

$$y = F(x) \quad \textbf{or} \quad x = G(y)$$

with F or G explicitly determined. However, it may still be possible to construct a local linear approximation at a specified point. For this exercise, specialize to the equation

$$x^2 + y^3 + xy - 11 = 0.$$

Assume without justification that the equation determines y as a function of x in some open rectangle that contains $(1, 2)$.

(a) Explain in plain English the meaning of "determines y as a function of x".

(b) Take the derivative with respect to x of both sides of the equation to determine the quantity $\left.\frac{dy}{dx}\right|_{(1,2)}$.

(c) Write a formula for the line L given by $L(x) = y'(1)(x - 1) + 2$.

(d) Use a computer to sketch the solution set to the equation as well as the line L. Describe any connection you notice between the two.

(e) The process in (b) is known as *implicit differentiation*. Explain the meaning of this word.

(f) How does this relate to the approach using paths that we used earlier?

Exercise 12

Suppose that you blow up a large spherical balloon so that the volume of air is increasing by a constant rate of 6 cubic inches per second.

(a) Determine the rate at which the radius of the balloon is changing when the radius is $\frac{1}{6}$ ft.

(b) Determine the rate at which the surface area is changing when the radius is $\frac{1}{6}$ ft.

Exercise 13

The implicit function theorem gives sufficient conditions to guarantee that one coordinate is a function of the other coordinate. A precise statement requires a new idea. Take f to be a real valued function that is defined in an open rectangle R in $\mathbb{R}^2$ and (a, b) in R.

(a) Explain the meaning of $f(a, \cdot)$ and $f(\cdot, b)$.

(b) Explain the meaning of $f'(a, \cdot)$ and $f'(\cdot, b)$.

(c) Explain the meaning of $f_y(a, \cdot)$ and $f_x(\cdot, b)$.

Take f to be the function given by

$$f(x, y) = x^2 + y^3 + xy.$$

(d) Write out $f(a, \cdot)$ and $f(\cdot, b)$ where $a = 1$ and $b = 2$.

(e) Write out $f'(a, \cdot)$ and $f'(\cdot, b)$ where $a = 1$ and $b = 2$.

(f) Write out $f_y(a, \cdot)$ and $f_x(\cdot, b)$ where $a = 1$ and $b = 2$.

Exercise 14

The implicit function theorem states the following:

For any real valued function f that is defined on an open rectangle D, and for any (x_0, y_0) in D, if

- f is continuous on D and $f(x_0, y_0) = 0$,
- f_y exists on D and is continuous at (x_0, y_0),
- and $f_y(x_0, y_0)$ is not equal to 0,

then there is an open rectangle $I \times J$ that contains (x_0, y_0) so that the set g that is given by

$$g = R \cap \{(x, y) \in D \colon f(x, y) = 0\}$$

is a function that is defined on I and differentiable at x_0.

State the analogous theorem with x and y switched.

Exercise 15

Take f to be the function given by

$$f(x, y) = x^2 + y^3 + xy.$$

For each of these choices of (a, b), determine whether the implicit function theorem guarantees that the level set

$$f(x, y) = 11$$

locally determines y as a function of x or x as a function of y in the intersection of some open rectangle that contains (a, b) :

(a) $(a, b) = (1, 2)$;

(b) $(a, b) = (-3, -1)$.