Linguistic Mapping

The Principles of Calculus I

VI

Local Linear Approximation

VI.1

Approximation by the Tangent Line

Classroom Exercises

Exercise 1

It is helpful to begin the study of the local linear approximation of functions in a "laboratory" of polynomial functions. The reason for this is that the order of intersection classifies the intersections of lines with polynomial functions. Classifying intersections is more complicated for more general functions.

With this in mind, take a to be in $\mathbb{R}$ and P to be the polynomial function given by

$$P(x) = x^3 + x^2 - 20x + 24.$$

(a) Expand the polynomial $P(x)$ as a polynomial in the variable $(x - a)$ by writing

$$P(x) = P(a + (x - a)).$$

(b) Use the expansion in (a) to write $P(x)$ as the sum

$$P(x) = L_a(x) + E_a(x),$$

where $L_a(x)$ is a line and the error E_a is $\mathrm{O}((x-a)^2)$. Be sure to explicitly write out an equation for E_a.

(c) Use the decomposition of P in (b) to facilitate the calculation of the derivative of P at a as the limit of a difference quotient.

(d) Explain in plain English why L_a is known as the local linear approximation of P at a.

Exercise 2

Extending the idea of the local linear approximation of a function to functions that are not polynomials requires only one new idea:

> A linear function L_a is a local linear approximation of f at $(a, f(a))$ if there is a function E_a so that
> $$f(x) = L_a(x) + E_a(x),$$
> where E_a is $\mathrm{o}(x-a)$.

(a) Justify that this definition for f is equivalent to there being a function η that is defined on the domain of f, continuous at a,

$$\eta(a) = 0, \quad \text{and} \quad f(x) = L_a(x) + \eta(x)(x-a).$$

(b) Suppose that f has a local linear approximation at a that is given by

$$L(x) = m(x - a) + f(a).$$

Use the limit definition of the derivative to compute $f'(a)$.

(c) Show that if f is defined on an interval I and differentiable at a point a in I, then f has a local linear approximation L_a at a that is given by

$$L_a(x) = f'(a)(x - a) + f(a).$$

(d) Show that if f is differentiable at a, then f is continuous at a. Is the converse true?

Exercise 3

Take f and g to be the functions given by

$$f(x) = |x|(x-1) \quad \text{and} \quad g(x) = x^{\frac{2}{3}}.$$

(a) Determine whether f and g are continuous at 0.

(b) Determine whether f and g are differentiable at 0.

Take f and g to be the functions given by

$$f(x) = |x|(x-1) \quad \textbf{and} \quad g(x) = x^{\frac{1}{3}}.$$

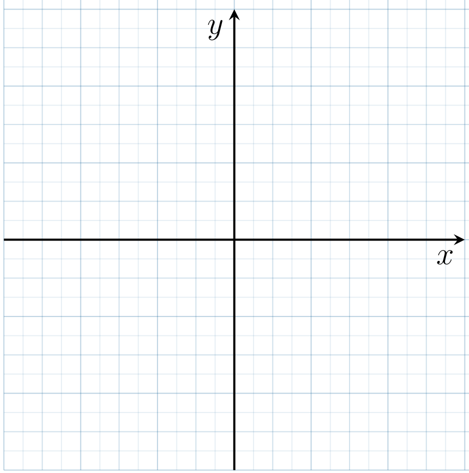
(c) Determine whether f and g have local linear approximations at 0.

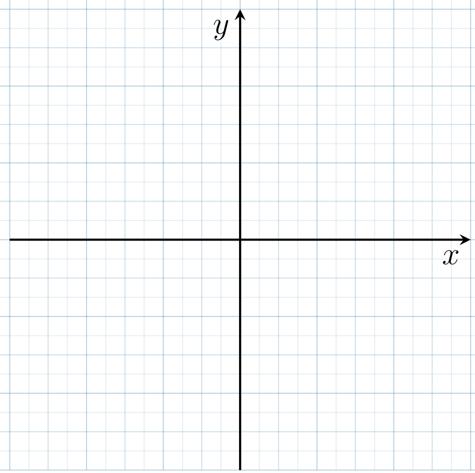
(d) Determine whether f and g have lines that are tangent at $(0, f(0))$ and $(0, g(0))$, respectively.

Take f and g to be the functions given by

$$f(x) = |x|(x-1) \quad \textbf{and} \quad g(x) = x^{\frac{1}{3}}.$$

(e) Sketch f and g. Identify any physical features involved in the answers in parts (a) - (d).





Exercise 4

Use the difference quotient definition of the derivative to establish at x_0 these local linear approximations:

- For any real numbers x and x_0, and any natural number n that is greater than 1,

$$x^n = \boxed{} + o(x - x_0);$$

Use the difference quotient definition of the derivative to establish at x_0 these local linear approximations:

- For any positive real numbers x and x_0,

$$\sqrt{x} = \boxed{} + \mathrm{o}(x - x_0);$$

Use the difference quotient definition of the derivative to establish at x_0 these local linear approximations:

- For any non-zero real numbers x and x_0,

$$\frac{1}{x} = \boxed{\phantom{\frac{1}{x_0} - \frac{1}{x_0^2}(x - x_0)}} + \mathrm{o}(x - x_0);$$

Use the difference quotient definition of the derivative to establish at x_0 these local linear approximations:

- For any a in $(0, 1) \cup (1, \infty)$ and any real numbers x and x_0,

$$a^x = \boxed{} + \mathrm{o}(x - x_0);$$

Use the difference quotient definition of the derivative to establish at x_0 these local linear approximations:

- For any real numbers x and x_0,

$$\sin(x) = \boxed{} + o(x - x_0);$$

$$\cos(x) = \boxed{} + o(x - x_0).$$

Exercise 5

For each of these choices of function f, decompose f into simpler functions whose local linear approximations you have already determined to determine the local linear approximation of f at x_0:

(a) $f(x) = 3\sqrt{x} + 7x^2$, $x_0 > 0$;

For each of these choices of function f, decompose f into simpler functions whose local linear approximations you have already determined to determine the local linear approximation of f at x_0:

(b) $f(x) = (x^3 + 4)\cos(x)$, any x_0 in $\mathbb{R}$;

For each of these choices of function f, decompose f into simpler functions whose local linear approximations you have already determined to determine the local linear approximation of f at x_0:

(c) $f(x) = x^{\frac{5}{2}}$, $x_0 > 0$. Hint: Use the equality $x^{\frac{5}{2}} = x^2\sqrt{x}$.

(d) In each of the above cases, identify a connection between the derivative of f and the derivatives of its factors and summands.

Exercise 6

Follow these steps to show that for any functions f and g that are defined on the same interval I with a in I and f and g differentiable at a, and for any real numbers A and B,

$$(Af + Bg)'(a) = Af'(a) + Bg'(a).$$

(a) Write down the local linear approximations for f and g.

(b) Use (a) to rewrite $Af(x) + Bg(x)$.

Follow these steps to show that for any functions f and g that are defined on the same interval I with a in I and f and g differentiable at a,

$$(fg)'(a) = f'(a)g(a) + f(a)g'(a).$$

(c) Use (a) to rewrite $f(x)g(x)$.

(d) Show that the resulting error terms have the correct local property.

Exercise 7

For each of these choices of function f, to determine the local linear linear approximation of f at x_0, decompose f into simpler functions whose local linear approximations you have already determined:

(a) $f(x) = \cos(x^2 + 3x + 1)$, for any x_0 in $\mathbb{R}$;

For each of these choices of function f, to determine the local linear linear approximation of f at x_0, decompose f into simpler functions whose local linear approximations you have already determined:

(b) $f(x) = \frac{1}{\sin(x)}$, for any x_0 with $\sin(x_0) \neq 0$;

For each of these choices of function f, to determine the local linear linear approximation of f at x_0, decompose f into simpler functions whose local linear approximations you have already determined:

(c) $f(x) = \sqrt{x^2 + \cos(x) + 1}$, for any x_0 in $\mathbb{R}$.

(d) In each of the above cases, identify a connection between the derivative of f and the derivative of its constituent parts.

Exercise 8

Follow these steps to show that for any functions f and g, if f is defined on an interval J, g is defined on an interval I, a is in I, $g(a)$ is in J, g is differentiable at a, and f is differentiable at $g(a)$, then

$$(f \circ g)'(a) = f'(g(a))g'(a).$$

(a) Write down the local linear approximations for f and g. It will help to explicitly write down the error terms as products.

(b) Use (a) to rewrite $(f \circ g)(x)$.

(c) Show that the resulting error term has the correct local property.

Exercise 9

For each of these choices of function f, decompose f to determine a formula for $f'(x)$:

(a) $f(x) = \sin(x)\sqrt{x^2 + \cos^2(x)}$;

(b) $f(x) = x\tan(x^2 + 1)$.

Exercise 10

For any function f that is differentiable at a point x_0 in its domain, the local linear approximation of f gives an important way of approximating f, namely, if h is a real number so that $x_0 + h$ is in $\mathcal{D}(f)$, then

$$f(x_0 + h) = f(x_0) + f'(x_0)h + \eta_f(x_0 + h)h, \quad \text{where} \quad \lim_{h \to 0} \eta_f(x_0 + h) = 0.$$

Approximations are only valuable if the error may be estimated, but for now we will ignore this fact and simply indicate that

$$f(x_0 + h) \approx f(x_0) + f'(x_0)h.$$

(a) Write down the local linear approximation for $\mathrm{pow}_{\frac{1}{3}}$ at $(8, 2)$.

(b) Use a suitable local linear approximation to approximate $(8.25)^{\frac{1}{3}}$.

(c) Determine the error of your approximation by using a calculator with a cube root function.

Exercise 11

A function is *continuously differentiable* on an interval $[a, b]$ if f is differentiable at every point in $[a, b]$ and the function f' is continuous on $[a, b]$.

Take f to be the function that is given by

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

(a) For any nonzero x, use the rules for differentiation to calculate $f'(x)$.

A function is *continuously differentiable* on an interval $[a, b]$ if f is differentiable at every point in $[a, b]$ and the function f' is continuous on $[a, b]$.

Take f to be the function that is given by

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

(b) Directly use the difference quotient definition of the derivative to determine $f'(0)$.

A function is *continuously differentiable* on an interval $[a, b]$ if f is differentiable at every point in $[a, b]$ and the function f' is continuous on $[a, b]$.

Take f to be the function that is given by

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

(c) Show that f is differentiable at 0, but f' is not continuous at 0.

The function f is, therefore, a function that is differentiable on $\mathbb{R}$, but not continuously differentiable on $\mathbb{R}$.