Linguistic Mapping

The Principles of Calculus II

V

Finite Approximation

V.9

Approximating Area in the Plane

Classroom Exercises

Exercise 1

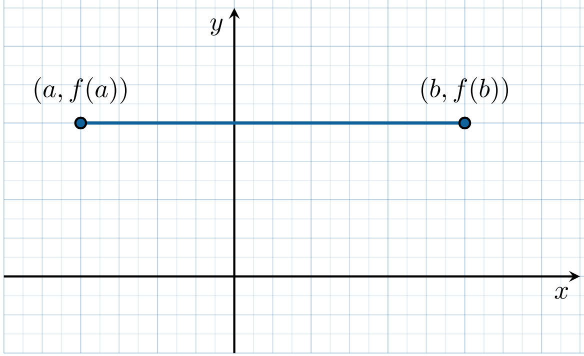
One way of motivating the Riemann integral of a bounded function f over a closed and bounded interval $[a, b]$ involves the finite approximation of motion. View f as the velocity of a particle that moves along the real number line. The goal is to reconstruct the particle's total displacement over the interval $[a, b]$ knowing only its velocity.

The simplest problem that we already know how to solve: f is constant.

(a) The velocity is non-negative and equal to V, determine the displacement.

(b) Sketch a picture that gives geometric meaning to this quantity.

(c) What is the geometric meaning if V is negative, and what is the physical meaning?



Exercise 2

One way of motivating the Riemann integral of a bounded function f over a closed and bounded interval $[a, b]$ involves the finite approximation of motion. View f as the velocity of a particle that moves along the real number line. The goal is to reconstruct the particle's total displacement over the interval $[a, b]$ knowing only its velocity.

A more involved problem that we still know how to solve: f is piecewise constant on m subintervals of $[a, b]$.

(a) Use a partition for $[a, b]$ to identify the m intervals on which f is constant. Precisely define the concept of a partition P for $[a, b]$.

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(b) Determine in terms of P the finite sequence of intervals $(I_P(n))$, the sequence of intervals that P defines. For each n, determine $|I_P(n)|$, the length of $I_P(n)$. Note that P cannot identify the openness or closedness of the intervals, but this does not matter for our purpose. At this point, take the intervals to be closed.

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A more involved problem that we still know how to solve: f is piecewise constant on m subintervals of $[a, b]$.

(c) Use additivity of displacement to reduce to part (a) and determine the total displacement over $[a, b]$.

Exercise 3

One way of motivating the Riemann integral of a bounded function f over a closed and bounded interval $[a, b]$ involves the finite approximation of motion. View f as the velocity of a particle that moves along the real number line. The goal is to reconstruct the particle's total displacement over the interval $[a, b]$ knowing only its velocity.

The inexact general problem: Replace f with a piecewise constant approximation that is constant on finitely many intervals.

(a) Given a partition P for $[a, b]$, use a tagging τ_P of P to sample f in order to determine the piecewise constant approximation. Precisely define the concept of a tagging for a partition P.

(b) Precisely define the concept of a tagged partition (P, τ_P) for $[a, b]$.

(c) Construct the piecewise constant approximation for f with the tagged partition (P, τ_P).

Note: It does not matter how you define this replacement of f at the endpoints of the intervals, so just make them left continuous when there is any ambiguity.

(d) Use Exercise 2 to identify a formula for the displacement $\mathcal{R}(f, P, \tau_P)$ associated to the piecewise constant approximation for f.

Exercise 4

One way of motivating the Riemann integral of a bounded function f over a closed and bounded interval $[a, b]$ involves the finite approximation of motion. View f as the velocity of a particle that moves along the real number line. The goal is to reconstruct the particle's total displacement over the interval $[a, b]$ knowing only its velocity.

The general problem: Approximate f by a sequence of piecewise constant approximations.

(a) Take (P_n) to be a sequence of partitions for $[a, b]$. For each n, precisely define what is meant by $\|P_n\|$, the *mesh* of P_n.

(b) Use a sequence of tagged partitions $((P_n, \tau_n))$ to identify a sequence of functions that approximates f.

(c) For the sequence of piecewise constant functions to approximate f, what type of sequence should $(\|P_n\|)$ be? Explain why.

(d) Suppose that the limit of the sequence $(\mathcal{R}(f, P_n, \tau_n))$ converges to a limit L. Identify a condition on f so that this limit L is meaningful without any other considerations.

(e) If f satisfies the condition you identified in (d), then f is Riemann integrable on $[a, b]$ and L is the Riemann integral of f on $[a, b]$. This quantity is often denoted by

$$L = \int_a^b f(x)\,\mathrm{d}x.$$

Note that physicists often write this quantity as

$$L = \int_a^b \mathrm{d}x\, f(x).$$

Identify a large class of functions that are Riemann integrable.

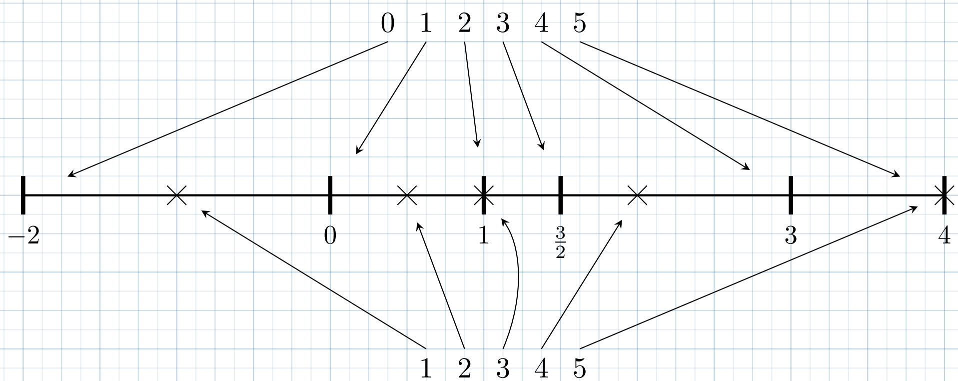
Exercise 5

Sketched below is a graphical representation of a partition P for the interval $[-2, 4]$, a tagging τ_P for P, and the finite sequence of intervals $(I_P(n))$ associated with P.

(a) Identify the domain of P and a formula for P.

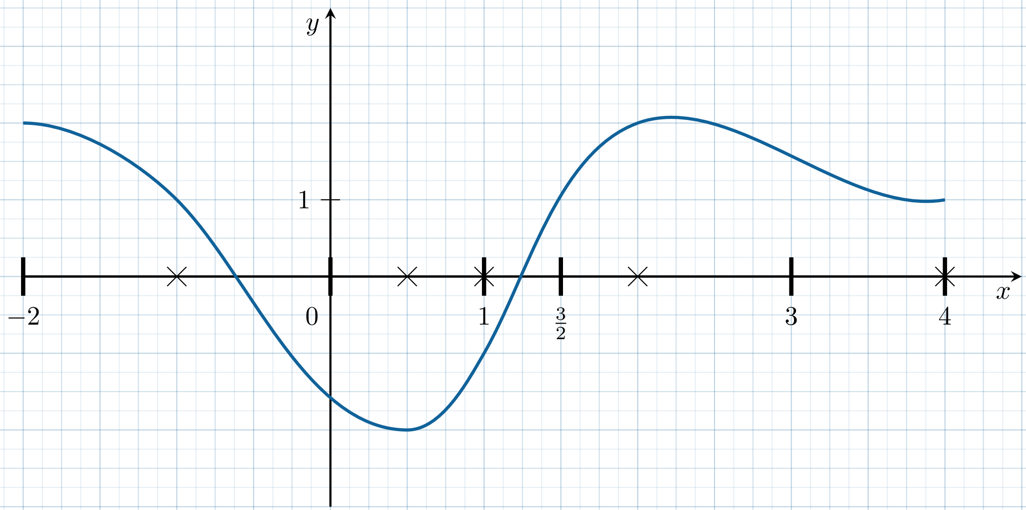
(b) Identify the domain of τ_P and a formula for τ_P.

(c) Explicitly write down the sequence $(I_P(n))$.



(d) Take f to be the function sketched below and give a graphical representation for the quantity $\mathcal{R}(f, P, \tau_P)$. Explain the meaning of this representation in terms of velocity.

(e) Write down a formula for $\mathcal{R}(f, P, \tau_P)$.

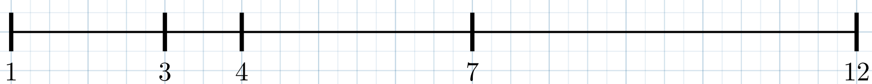


Take P to be the partition for $[1, 12]$ that is given by

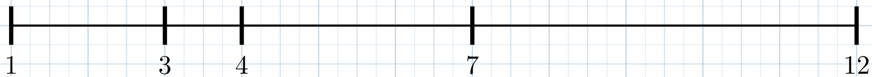
$$P(0) = 1,\ P(1) = 3,\ P(2) = 4,\ P(3) = 7, \quad \text{and} \quad P(4) = 12.$$

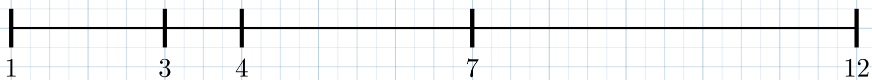
(a) Determine $\|P\|$, the mesh of P.

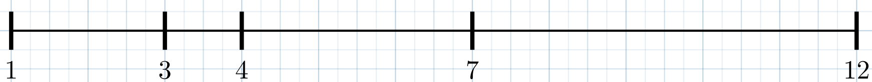
(b) Determine the finite sequence of intervals $(I_P(n))$, the sequence of intervals that P defines. For each n, determine $|I_P(n)|$, the length of $I_P(n)$.



(c) Identify a left endpoint τ_L, a right endpoint, τ_R, and midpoint tagging τ_M for the partition P.







Exercise 8

For any sequence of tagged partitions (P_n, τ_n) of $[0,1]$ so that $(\|P_n\|)$ is a null sequence,

$$\lim_{n\to\infty} \mathcal{R}(\mathrm{pow}_2, P_n, \tau_n) = \int_0^1 x^2 \,\mathrm{d}x.$$

Since pow_2 is continuous, this limit exists and is independent of choice of sequence of tagged partitions.

(a) For each n, take P_n to be an even partition of $[0,1]$ with n intervals, τ_n to be a left endpoint tagging for P_n, and τ_n^* to be a right endpoint tagging for P_n. Write down for each n a formula for $\mathcal{R}(\mathrm{pow}_2, P_n, \tau_n)$ and $\mathcal{R}(\mathrm{pow}_2, P_n, \tau_n^*)$.

For any sequence of tagged partitions (P_n, τ_n) of $[0, 1]$ so that $(\|P_n\|)$ is a null sequence,

$$\lim_{n\to\infty} \mathcal{R}(\mathrm{pow}_2, P_n, \tau_n) = \int_0^1 x^2 \,\mathrm{d}x.$$

Since pow_2 is continuous, this limit exists and is independent of choice of sequence of tagged partitions.

(b) Compute the quantity $\lim_{n\to\infty} \mathcal{R}(\mathrm{pow}_2, P_n, \tau_n)$ and $\lim_{n\to\infty} \mathcal{R}(\mathrm{pow}_2, P_n, \tau_n^*)$.

(c) Simulate the above sequences of sums and their differences. Link

Exercise 9

Take f to be Riemann integrable on $[-1, 4]$. Use the equalities

$$\int_{-1}^{1} f(x)\,\mathrm{d}x = 3 \quad \textbf{and} \quad \int_{1}^{4} f(x)\,\mathrm{d}x = 7$$

to evaluate

$$\int_{-1}^{4} f(x)\,\mathrm{d}x.$$

Exercise 10

Take f and g to be Riemann integrable on $[3, 5]$. Use the equalities

$$\int_3^5 f(x)\,\mathrm{d}x = 2 \quad \textbf{and} \quad \int_3^5 g(x)\,\mathrm{d}x = -1$$

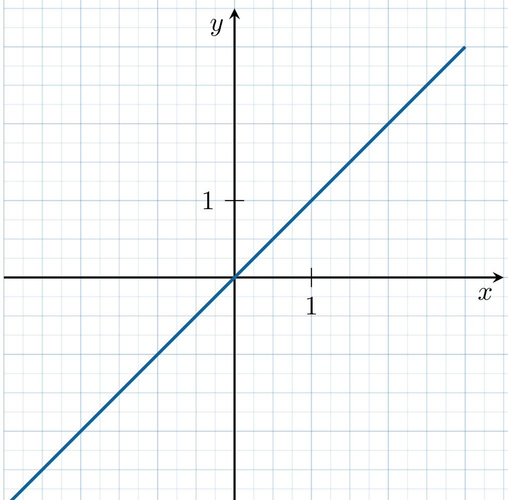
to evaluate

$$\int_3^5 (7f(x) + 2g(x))\,\mathrm{d}x.$$

Exercise 11

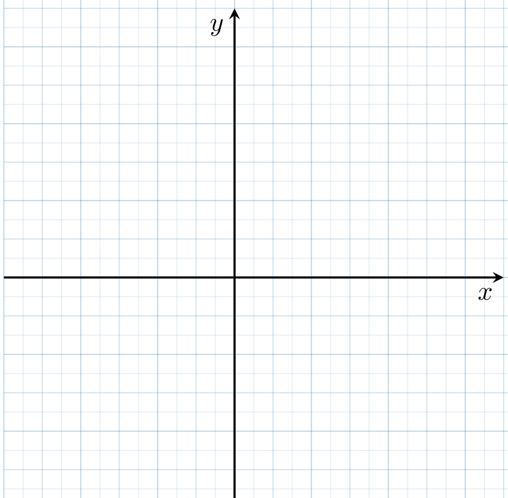
Use symmetry, antisymmetry, or basic geometry to evaluate these integrals:

(a) $\displaystyle\int_{-2}^{3} x \, \mathrm{d}x$;



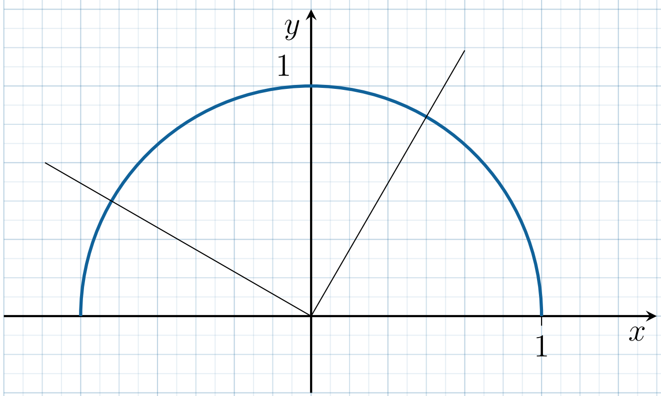
Use symmetry, antisymmetry, or basic geometry to evaluate these integrals:

(b) $\displaystyle\int_{-1}^{1} x^{\frac{5}{3}}\,\mathrm{d}x$;



Use symmetry, antisymmetry, or basic geometry to evaluate these integrals:

(c) $\displaystyle\int_{-\frac{\sqrt{3}}{2}}^{\frac{1}{2}} \sqrt{1-x^2}\,\mathrm{d}x.$



Exercise 12

For any function f, if f is continuous on $[a, b]$, then f is Riemann integrable on $[a, b]$.
Take P to be any partition of $[a, b]$.

(a) Explain what it means for τ_m to be a minimal tagging for $[a, b]$ with respect to f.

(b) Explain what it means for τ_M to be a maximal tagging for $[a, b]$ with respect to f.

For any function f, if f is continuous on $[a,b]$, then f is Riemann integrable on $[a,b]$. Take P to be any partition of $[a,b]$.

(c) Take ω_f to be a modulus of continuity for f on $[a,b]$ and use ω_f to provide an upper bound for $f(\tau_M(i)) - f(\tau_m(i))$ for any i in $\mathcal{D}(P) \setminus \{0\}$.

For any function f, if f is continuous on $[a, b]$, then f is Riemann integrable on $[a, b]$.
Take P to be any partition of $[a, b]$.

(d) Use the fact that

$$\mathcal{R}(f, P, \tau_m) \leq \int_a^b f(x)\,\mathrm{d}x \leq \mathcal{R}(f, P, \tau_M)$$

to estimate the maximum possible error for a Riemann sum approximation of

$$\int_a^b f(x)\,\mathrm{d}x$$

that uses a partition P with mesh $\|P\|$.

Exercise 13

For each of these choices of interval $[a, b]$ and positive real number δ, estimate the maximum possible error for a Riemann sum approximation of

$$\int_a^b \frac{1}{x}\,\mathrm{d}x$$

that uses a partition P with mesh equal to δ:

(a) $[a, b] = [1, 4]$ and $\delta = \frac{1}{4}$;

(b) $[a, b] = [4, 8]$ and $\delta = \frac{1}{4}$;

(c) $[a, b] = [\frac{1}{100}, 1]$ and $\delta = \frac{1}{10}$.