Linguistic Mapping

The Principles of Calculus II

V

Finite Approximation

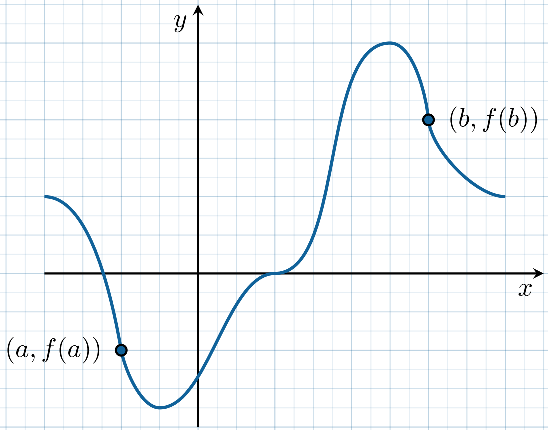
V.8

Approximating Change

Classroom Exercises

For any non-empty interval $[a,b]$ and any real-valued function f on $[a,b]$, precisely define the meaning of $f^{\text{ave}}([a,b])$, the average rate of change of f on $[a,b]$.

(a) Visualize the meaning of this definition on the provided graph.



(b) Given that the argument of f describes a physical quantity with physical unit u_1 and the values that f takes on describes a physical quantity with physical unit u_2, determine the physical units for the average rate of change of f on $[a, b]$.

(c) Given that f describes for each point in time the position of a particle that moves along the real number line, determine the physical units of the average rate of change of f on any non-empty interval in the domain of f.

(d) Given the assumptions in (c), identify the physical interpretation of the average rate of change of f on $[a, b]$.

Exercise 2

For each of these choices of function f and interval $[a, b]$, determine $f^{\text{ave}}([a, b])$, the average rate of change of f on $[a, b]$:

(a) $f(x) = 1$, $[a, b] = [2, 7]$;

(b) $f(x) = 2x + 5$, $[a, b] = [-1, 2]$;

(c) $f(x) = x^3 + 5x^2 + 1$, $[a, b] = [1, 3]$.

Exercise 3

For any non-empty interval $[a, b]$ and any path c on $[a, b]$ that is valued in $\mathbb{R}^2$ (or $\mathbb{R}^3$), precisely define the meaning of the average rate of change of c on $[a, b]$.

(a) Identify the type of quantity that is the average rate of change of c on $[a, b]$.
(Example: A real number, a point in the plane, etc.)

(b) Given that the argument of c describes a physical quantity with physical unit u_1 and the values that c takes on describes a physical quantity with physical unit u_2, determine the physical units for the average rate of change of c on $[a, b]$.

(c) Given that c describes for each point in time the position of a particle that moves in the plane, determine the physical units of the average rate of change of c on any non-empty interval in the domain of c.

(d) Given the assumptions in (c), identify the physical interpretation of the average rate of change of c on $[a, b]$.

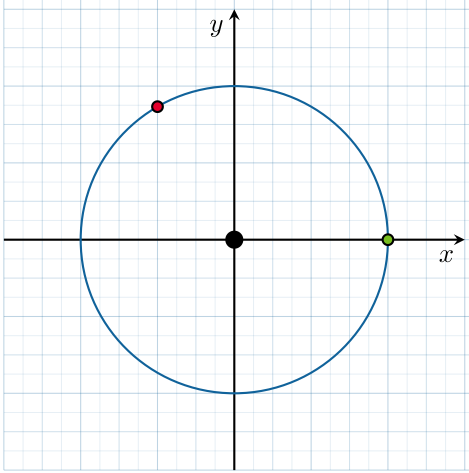
Exercise 4

Take c to be the path that is given by

$$c(t) = (\cos(t), \sin(t)).$$

(a) For any real number a in $(0, 2\pi]$, determine the average velocity and the average speed of c on the interval $[0, a]$.

(b) Simulate the motion determined by c on $[0, a]$, as well as the constant velocity motion on the intervals $[0, a]$, where the constant velocity is the average velocity over $[0, a]$ and the path describes a particle that is at $(1, 0)$ at time 0.



Exercise 5

For any point x_0, and any non-zero real number h, take $f_{x_0}^{\text{ave}}(h)$ to be the average value of the function f on $[x_0, x_0 + h]$ if h is positive and on $[x_0 + h, x_0]$ if h is negative.

(a) Take f to be the function pow_2 and for any x_0, identify a formula for $f_{x_0}^{\text{ave}}(h)$.

(b) Take x_0 and h to be sliders, and sketch f using a computer together with the line with slope $f_{x_0}^{\text{ave}}(h)$ that passes through $(x_0, f(x_0))$. Vary x_0 and h.

For any point x_0, and any non-zero real number h, take $f_{x_0}^{\text{ave}}(h)$ to be the average value of the function f on $[x_0, x_0 + h]$ if h is positive and on $[x_0 + h, x_0]$ if h is negative.

(c) Determine the unique continuous extension of $f_{x_0}^{\text{ave}}$ to 0.

Exercise 6

Recall that for any function f that is defined on an interval I that contains more than one point and for any x_0 in I, the quantity $f'(x_0)$ is defined by the limit

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

For each of these choices of function f, use this definition to directly calculate $f'(x)$ for every x where this limit exists:

(a) $f(x) = 1$;

Note: The function f' is the derivative of f and f is said to be differentiable at every point at which f' is defined.

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$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

For each of these choices of function f, use this definition to directly calculate $f'(x)$ for every x where this limit exists:

(b) $f(x) = x^n$ for $n \geq 2$;

Recall that for any function f that is defined on an interval I that contains more than one point and for any x_0 in I, the quantity $f'(x_0)$ is defined by the limit

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

For each of these choices of function f, use this definition to directly calculate $f'(x)$ for every x where this limit exists:

(c) $f(x) = \frac{1}{x}$;

Recall that for any function f that is defined on an interval I that contains more than one point and for any x_0 in I, the quantity $f'(x_0)$ is defined by the limit

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

For each of these choices of function f, use this definition to directly calculate $f'(x)$ for every x where this limit exists:

(d) $f(x) = \frac{x-1}{x^2+5}$.

Exercise 7

For each of these choices of function f, use limits to calculate $f'(x)$ for every x where this limit exists:

(a) $f(x) = \sin(x)$;

For each of these choices of function f, use limits to calculate $f'(x)$ for every x where this limit exists:

(b) $f(x) = x\sin(3x)$;

Exercise 8

For each of these choices of function f, use limits to calculate $f'(x)$ for every x where this limit exists:

(c) $f(x) = \sqrt{3x+1}$.

Recall that for any path c that is defined on an interval I that contains more than one point and for any t_0 in I, the quantity $c'(t_0)$ is defined by the limit

$$c'(t_0) = \lim_{h \to 0} \frac{c(t_0 + h) - c(t_0)}{h}.$$

There are functions x and y so that for each t,

$$c(t) = (x(t), y(t)), \quad \text{and so} \quad c'(t_0) = \langle x'(t), y'(t) \rangle$$

for each t_0 for which the limits exist.

Determine $c'(t)$ for each of these choices of path c and note that derivatives of spatial paths are similarly defined:

(a) $c(t) = (t^2, t^3)$;

Recall that for any path c that is defined on an interval I that contains more than one point and for any t_0 in I, the quantity $c'(t_0)$ is defined by the limit

$$c'(t_0) = \lim_{h \to 0} \frac{c(t_0 + h) - c(t_0)}{h}.$$

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for each t_0 for which the limits exist.

Determine $c'(t)$ for each of these choices of path c and note that derivatives of spatial paths are similarly defined:

(b) $c(t) = (\cos(t), \sin(t), t)$.

Exercise 10

Take c to be the path that is given by

$$c(t) = (t, f(t)), \quad \textbf{where} \quad f(t) = t^2 + t + 1.$$

(a) Determine the limit $v(t_0)$, where

$$v(t_0) = \lim_{h \to 0^+} \frac{c(t_0 + h) - c(t_0)}{\|c(t_0 + h) - c(t_0)\|}.$$

Take c to be the path that is given by

$$c(t) = (t, f(t)), \quad \textbf{where} \quad f(t) = t^2 + t + 1.$$

(b) Determine the equation of the line L that passes through $c(t_0)$ and along which the vector $v(t_0)$ moves points.

(c) Calculate $f'(t_0)$ by taking a limit and compare this quantity with the slope of L.

Exercise 11

Recall that for each x in $\mathbb{R}$, the function $\exp$ is defined by the infinite series

$$\exp(x) = 1 + x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + \cdots .$$

It turns out that the function $\exp$ is alternatively given by the formula

$$\exp(x) = \mathrm{e}^x$$

and is, in fact, the base e exponential function. It is an increasing continuous function with domain equal to $\mathbb{R}$, range equal to $(0, \infty)$, and it is differentiable with

$$\exp'(x) = \exp(x).$$

(a) Use the ratio test to show that for each x, $\exp(x)$ is absolutely convergent.

(b) Show that if a is a non-zero real number and f is the function that is given by

$$f(x) = \exp(ax), \quad \text{then} \quad f'(x) = a\exp(ax).$$

(c) Determine $\exp_b'(x)$ for any b in $(0,1) \cup (1,\infty)$.

Exercise 12

Use $\exp$ to define for any real number r the function pow_r and argue that pow_r is a continuous function on $[0, \infty)$. Hint: Is the inverse function $\ln$ a continuous function? How might you indirectly establish this fact?