Linguistic Mapping

The Principles of Calculus I

V

Finite Approximation

V.6

Continuous Functions

Classroom Exercises

Exercise 1

For any real-valued function f defined on a subset of $\mathbb{R}$, f is continuous at a point x_0 in $\mathcal{D}(f)$ means that for any sequence (x_n, y_n) in $\mathbb{R}^2$, if (x_n, y_n) is in f and (x_n) converges to x_0, then (y_n) converges to $f(x_0)$.

(a) Explain what this statement means in plain English.

For any real-valued function f defined on a subset of $\mathbb{R}$, f is continuous at a point x_0 in $\mathcal{D}(f)$ means that for any sequence (x_n, y_n) in $\mathbb{R}^2$, if (x_n, y_n) is in f and (x_n) converges to x_0, then (y_n) converges to $f(x_0)$.

(b) Restate the definition of continuity in terms of left and right limits of functions.

(c) State what it means for f to be continuous on a subset S of $\mathbb{R}$.

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(d) Take f to be the function that is given by

$$f(x) = \begin{cases} 2x + a & \text{if } x < 1 \\ x^2 & \text{if } 1 \leq x \leq b \\ x + 6 & \text{if } x > b. \end{cases}$$

Identify real numbers a and b so that f is continuous.

Exercise 2

Take f and g to be the functions that are given by

$$f(x) = \begin{cases} 2x - 7 & \text{if } x \neq 5 \\ -4 & \text{if } x = 5 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 5x & \text{if } x \leq 1 \\ x + 4 & \text{if } x > 1. \end{cases}$$

(a) Determine the limits

$$\lim_{x \to 1} g(x), \quad \lim_{x \to 5} f(x), \quad \text{and} \quad \lim_{x \to 1} (f \circ g)(x).$$

(b) Determine the validity of the statement

$$\lim_{x \to 1} f(g(x)) = f\left(\lim_{x \to 1} g(x)\right).$$

(c) Change how f is defined at 5 so that the equality in (b) is valid.

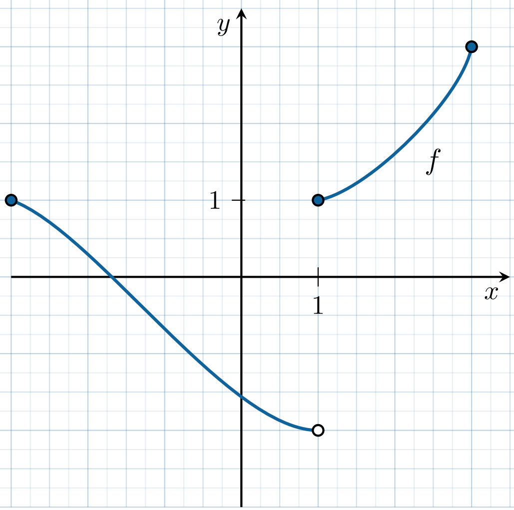
Exercise 3

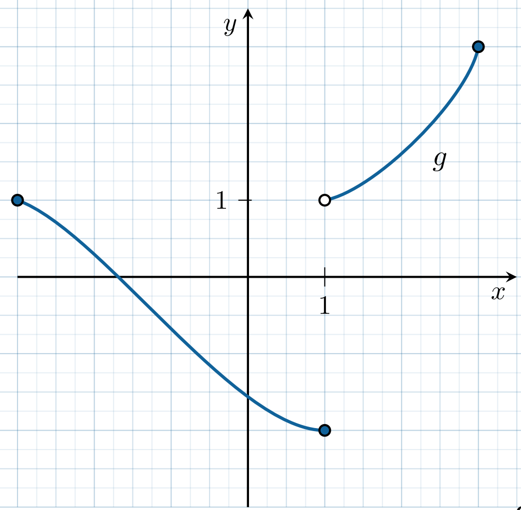
Just as there is a notion of a left and right limit of a function, there is a notion of left and right continuity of a function at a point x_0.

(a) State what it means for a function to be *left continuous* at x_0.

(b) State what it means for a function to be *right continuous* at x_0.

(c) Given the sketches of the function f and g below, identify the directional continuity of f and g at 1.





Exercise 4

Take a, b, c, and d to be functions that are defined in an open interval that contains 2 and

$$\lim_{x\to 2} a(x) = 1, \quad \lim_{x\to 2} b(x) = 3, \quad \lim_{x\to 2} c(x) = -4, \quad \textbf{and} \quad \lim_{x\to 2} d(x) = 7.$$

It turns out that for any real number r, pow_r is continuous on its domain. The trigonometric functions are also continuous on their domains, as is the reciprocal function. Use these facts to determine the following limits:

(a) $\lim\limits_{x\to 2} \big(4d(x) - a(x)\big)^{\frac{2}{3}}$;

(b) $\lim\limits_{x\to 2} \big(a(x)d(x) + \sin(4b(x) + 3c(x))\big)$;

(c) $\lim\limits_{x\to 2} \frac{1}{3+\cos(\pi a(x))}$.

Exercise 5

The idea of a continuous extension $\tilde{f}$ of a function f plays a crucial role in our subject.

(a) State precisely what it means for $\tilde{f}$ to be a continuous extension of a function f.

(b) Assuming that $\tilde{f}$ exists, identify a condition on $\mathcal{D}(\tilde{f})$ that guarantees that $\tilde{f}$ is the unique continuous extension of f to $\mathcal{D}(\tilde{f})$.

(c) Take f to be the function that is given by

$$f(x) = \frac{\sin(x)}{x}.$$

Determine the maximal domain of f and identify a continuous extension of f to all of $\mathbb{R}$. The continuous extension of f is the function sinc (read: "sink").

(d) Use the continuity of $\sin$ to determine that sinc is continuous.

(e) In general, must the continuous extension of a function be continuous?

Your friend makes the following comment:

"The limit from the left at 0 of the square root function does not exist because the square root function is not defined to the left of 0. Therefore, the square root function is not continuous at 0."

Identify your friend's misunderstanding and explain why the square root function is continuous at 0.

Exercise 7

True or False: Every real-valued function with domain $\{1, 3, 10\}$ is continuous.

Exercise 8

The *Extremal Value Theorem* is an important theorem in our subject.

(a) The theorem involves certain kinds of real valued functions defined on certain kinds of intervals. What kind of functions and what kind of intervals?

(b) Precisely state the Extremal Value Theorem.

(c) Identify counterexamples of the statement of theorem with any of the hypotheses weakened.

Exercise 9

To say that a real-valued function f that is defined on a subset D of $\mathbb{R}$ has the *intermediate value property* means that for any real numbers a and b in D with a less than b, if y_0 is a value that is between $f(a)$ and $f(b)$, then there is c in $D \cap (a, b)$ so that $f(c)$ is equal to y_0.

(a) Precisely state the *Intermediate Value Theorem*.

(b) Use the bisection method to approximate the value $\sqrt{8}$ to within a value of $\frac{1}{10}$. Be sure to carefully state where you use of the intermediate value theorem in your application of the bisection method.

(c) The following statement summarizes both the Intermediate Value theorem and the Extremal Value Theorem:

> Continuous functions take closed and bounded intervals to closed and bounded intervals.

Explain why this summary is valid and why both theorems are captured by this statement.

Exercise 10

For any path c defined on a subset of $\mathbb{R}$, c is continuous at a point t_0 in $\mathcal{D}(c)$ means that for any sequence $(t_n, (x_n, y_n))$ in $\mathbb{R} \times \mathbb{R}^2$, if $(t_n, (x_n, y_n))$ is in c and (t_n) converges to t_0, then $\big(\|(x_n, y_n) - c(t_0)\|\big)$ is a null sequence.

(a) Explain what this statement means in plain English.

For any path c defined on a subset of $\mathbb{R}$, c is continuous at a point t_0 in $\mathcal{D}(c)$ means that for any sequence $(t_n, (x_n, y_n))$ in $\mathbb{R} \times \mathbb{R}^2$, if $(t_n, (x_n, y_n))$ is in c and (t_n) converges to t_0, then $\big(\|(x_n, y_n) - c(t_0)\|\big)$ is a null sequence.

(b) Explain why the definition is equivalent to the definition that states that a path from $\mathbb{R}$ to $\mathbb{R}^2$ is continuous if it has continuous component functions.

(c) Take c to be the path that is given by

$$c(t) = \begin{cases} (t-1, 3t) & \text{if } t < 2 \\ (3t-5, t^2) & \text{if } t \geq 2. \end{cases}$$

Determine whether c is continuous and, if not, where it fails to be continuous.

Exercise 11

Use the $\arctan$ function to construct paths with the following properties:

(a) The path c is continuous and describes the position of a particle that moves to the right on the line segment from $(1, 3)$ to $(5, 6)$, is at $(1, 3)$ at time 0, is never at the same point at different time points, never reaches $(5, 6)$, but has the property that

$$\lim_{t\to\infty} \|(5, 6) - c(t)\| = 0.$$

(b) The path c is continuous and describes the position of a particle that moves counterclockwise around a circle of radius 2 that is centered at $(3, 7)$, that is at $(5, 7)$ at time 0, is never at the same point at different time points, and gets as close as we like to $(1, 7)$, but never reaches $(1, 7)$.