Linguistic Mapping

The Principles of Calculus II

V

Finite Approximation

V.4

Summation

Classroom Exercises

Exercise 1

The interval $[0,2]$ may be decomposed in stages in the following way:

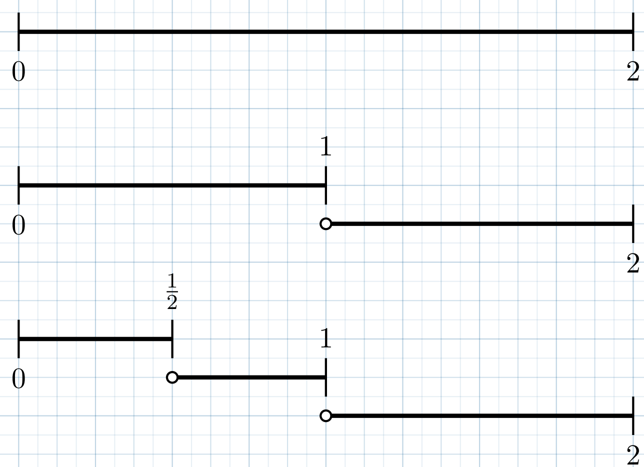
(a) $[0,2]=[0,1]\cup(1,2]$;

(b) $[0,2]=[0,\frac{1}{2})\cup[\frac{1}{2},1]\cup(1,2]$;

(c) $[0,2]=[0,\frac{1}{4})\cup[\frac{1}{4},\frac{1}{2})\cup[\frac{1}{2},1]\cup(1,2]$;

(d) $[0,2]=[0,\frac{1}{8})\cup[\frac{1}{8},\frac{1}{4})\cup[\frac{1}{4},\frac{1}{2})\cup[\frac{1}{2},1]\cup(1,2]$.

Compute the length of $[0,2]$ by computing the length of each interval in the given decomposition.



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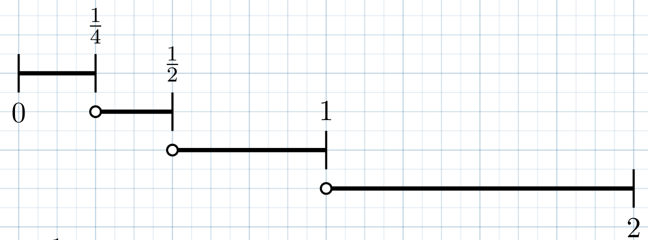

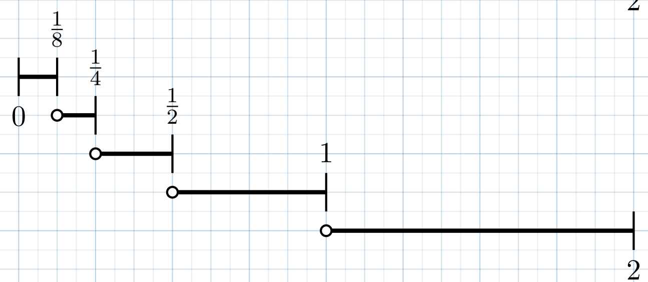

Exercise 2

Continue the pattern of decomposing $[0,2]$ from the previous problem to obtain a "sum"

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots+\frac{1}{2^n}+\cdots.$$

(a) Use recursion to identify a sequence (S_n) that gives for each n the sum of the first n summands of this "sum".

(b) Determine the difference between the length of the interval $[0,2]$ and the sum of the first n summands of this "sum".

(c) Identify the limit of the sequence (S_n).

(d) Why is the word sum used above in quotation marks?

Exercise 3

The idea of a recursive sequence can make precise the idea of the summation of an infinite number of summands. Take (a_n) to be any sequence in $\mathbb{R}$ and make a recursive definition that captures in terms of some sequence (S_n) the meaning of the infinite sum

$$a_1 + a_2 + \cdots + a_n + \cdots .$$

(a) Identify the meaning of the symbol $\sum_{m=1}^{n} a_m$. How do you read this symbol?

(b) For each natural number N and each natural number k, identify the meaning of the symbol $\sum_{m=N}^{N+k} a_m$. How do you read this symbol?

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$$a_1 + a_2 + \cdots + a_n + \cdots .$$

(c) Identify the meaning of the symbol $\sum_{m=1}^{\infty} a_m$. How do you read this symbol? Be careful, it can be used to mean two different things.

(d) Identify the meaning of the symbol $\sum a_m$. How do you read this symbol? Be careful, it can be used to mean two different things.

(e) Do you see a similarity in the dual meaning of the symbol given in (c) and (d) and in any dual meaning when we write, for example, $1 + 3$?

Exercise 4

The *axiom of induction* is a fundamental statement about natural numbers states that for any set S, if 1 is in S and for any natural number n, $n+1$ is in S, then S contains the natural numbers.

(a) Explain how one can use this axiom to prove statements that involve natural numbers.

(b) Use the axiom of induction to prove that

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

(c) Directly determine the above sum. Hint:

$$2(1 + 2 + \cdots + n) = (1 + 2 + \cdots + n) + (n + (n-1) + \cdots + 1).$$

Exercise 5

For any real numbers a and b, and any natural numbers m and n determine a formula for the sum

$$\sum_{k=m}^{m+n}(ak+b).$$

Exercise 6

For any real number r, take $(P_r(n))$ to be the sequence that is recursively defined by

$$\begin{cases} P_r(0) = 1 \\ P_r(n+1) = 1 + rP_r(n). \end{cases}$$

(a) Write down the first 5 terms of this sequence.

(b) Show that this is the same sequence that is given by

$$\begin{cases} P_r(0) = 1 \\ P_r(n+1) = P_r(n) + r^{n+1}. \end{cases}$$

Hint: The key is to use induction on n.

For any real number r, take $(P_r(n))$ to be the sequence that is recursively defined by

$$\begin{cases} P_r(0) = 1 \\ P_r(n+1) = 1 + rP_r(n) \end{cases} \quad \text{or equivalently by} \quad \begin{cases} P_r(0) = 1 \\ P_r(n+1) = P_r(n) + r^{n+1}. \end{cases}$$

(c) Use the two recursive definitions for $P_r(n+1)$ to identify for any n a formula for $P_r(n)$.

(d) Show that the sequence of partial sums is convergent if r is in $(-1, 1)$, and determine its limit.

Exercise 7

Very few infinite series can be precisely evaluated. Telescoping series are a notable example of such series.

(a) In plain English, describe what it means for an infinite series to be a telescoping series.

What follows are two examples of telescoping series. Compute each by writing each infinite series explicitly as a sequence of partial sums. We utilize two different notations for the infinite series to help to familiarize you with both.

(b) $\sum_{n=1}^{\infty} \frac{2}{(n+1)(n+2)}$;

(c) $\sum \frac{1}{n(n+3)}$.

Exercise 8

The main result that we have about the convergence of sequences is the *monotone convergence theorem for sequences*.

(a) State this theorem.

(b) Use this theorem to show that $\sum \frac{1}{n^2}$ is convergent. Hint: Can you find an upper bound for this series by comparing it to a telescoping series that you can more easily evaluate?

(c) The monotone convergence theorem for sequences implies the *comparison test for the convergence of series*. State the comparison test and explain how the monotone convergence theorem gives rise to this test.

Use the comparison test to determine the convergence or divergence of each of these series:

(a) $\sum \frac{n+\sqrt{n}}{n^3+5}$;

(b) $\sum \frac{n\sqrt{n}}{n^2+1}$;

(c) $\sum \frac{n^3+n+1}{n^4+2n-1}$;

(d) $\sum \frac{n^3+1}{2^n}$;

(e) $\sum \frac{n^2+1}{n!}$.

Try to use other convergence tests as well, in particular, the limit comparison test for (a), (b), and (c), and the ratio test for (d) and (e).

Exercise 10

Alternating series form an important type of series. Take (S_n) to be an infinite series and (a_m) to be its underlying sequence.

(a) What does it mean for $\sum a_m$ (or equivalently (S_n)) to be an alternating series.

(b) State the *alternating series test* and explain why it is true.

Alternating series form an important type of series. Take (S_n) to be an infinite series and (a_m) to be its underlying sequence.

(c) Given any bijection f from $\mathbb{N}$ to $\mathbb{N}$, if (S_n) is alternating and convergent and if (T_n) is an infinite series with $(a_{f(m)})$ as its underlying sequence, must (T_n) also be convergent? If it is convergent, must (S_n) and (T_n) have the same limit?

(d) What does it mean for a series to be *absolutely convergent*?

(e) Under what conditions is (d) guaranteed to be true?

Exercise 11

It is helpful to utilize the term-by-term comparison of series to demonstrate the divergence to ∞ or $-\infty$ of a series. Given that $\sum \frac{1}{n}$ diverges to ∞ and that $\sum \frac{1}{n^2}$ is convergent, determine the absolute convergence, convergence, and divergence of the following series:

(a) $\sum \frac{(-1)^n}{\sqrt{n}}$;

(b) $\sum \frac{(-1)^n(n+1)}{n^3+2n+1}$;

(c) $\sum \frac{(-1)^n}{2n+1}$;

(d) $\sum \frac{2n+1}{n+4}$.