Linguistic Mapping

The Principles of Calculus II

V

Finite Approximation

V.3

Measurement of a Circle

Classroom Exercises

Exercise 1

To show that the path c that is given by

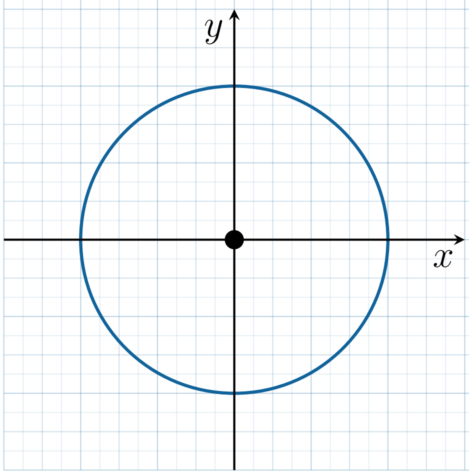
$$c(t) = \begin{cases} \left(1-t, \sqrt{1-(1-t)^2}\right) & \text{if } 0 \leq t \leq 2 \\ \left(t-3, -\sqrt{1-(t-3)^2}\right) & \text{if } 2 < t \leq 4 \end{cases}$$

parameterizes the unit circle, follow these steps:

(a) Show that c is one-to-one;

(b) For every p in $\mathscr{C}$, show that there is a t so that $c(t)$ is equal to p.

Explain how c determines a counterclockwise ordering of the points of $\mathscr{C}$.



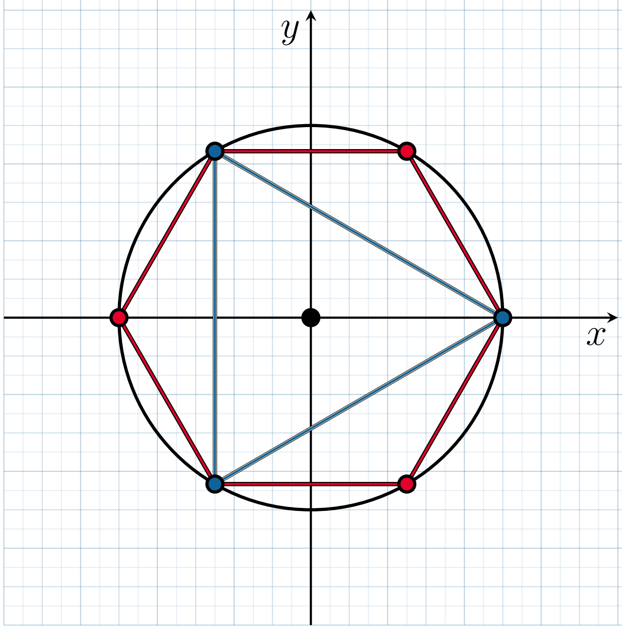
Exercise 2

Explain what it means, from an algebraic perspective, for the fraction of $\mathscr{C}$ associated to a point p in $\mathscr{C}$, the quantity $\mathrm{Frac}(p)$, to be equal to $\frac{1}{3}$.

(a) Identify the point p with this property.

(b) Identify a point q with the property that $\mathrm{Frac}(q)$ is $\frac{2}{3}$.

(c) Identify a point r with the property that $\mathrm{Frac}(r)$ is $\frac{1}{6}$.



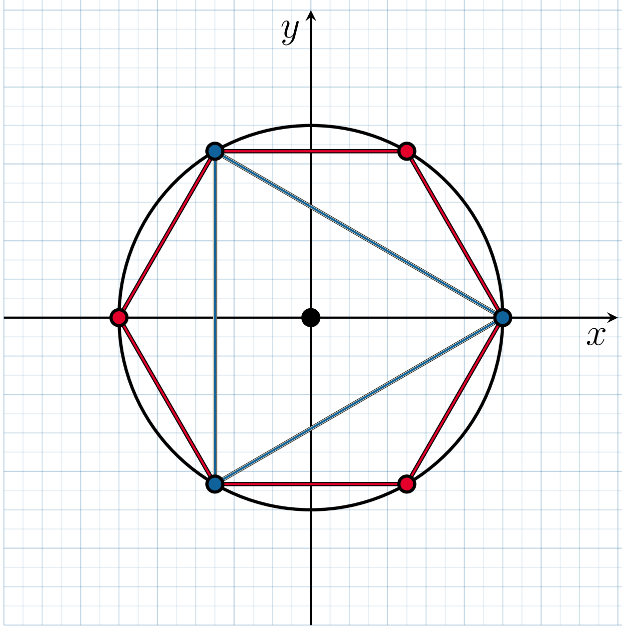
Exercise 3

For any natural number n and any natural number m in $[0, 3 \cdot 2^n)$, explain how to find a real number t_0 so that $\operatorname{Frac} c(t_0)$ is equal to $\frac{m}{3 \cdot 2^n}$.

(a) First identify a procedure for determining the coordinates of $c(t_0)$.

(b) Second, identify how to determine t_0.

(c) Use this procedure to determine t_0 so that $\operatorname{Frac} c(t_0)$ is equal to $\frac{5}{6}$.



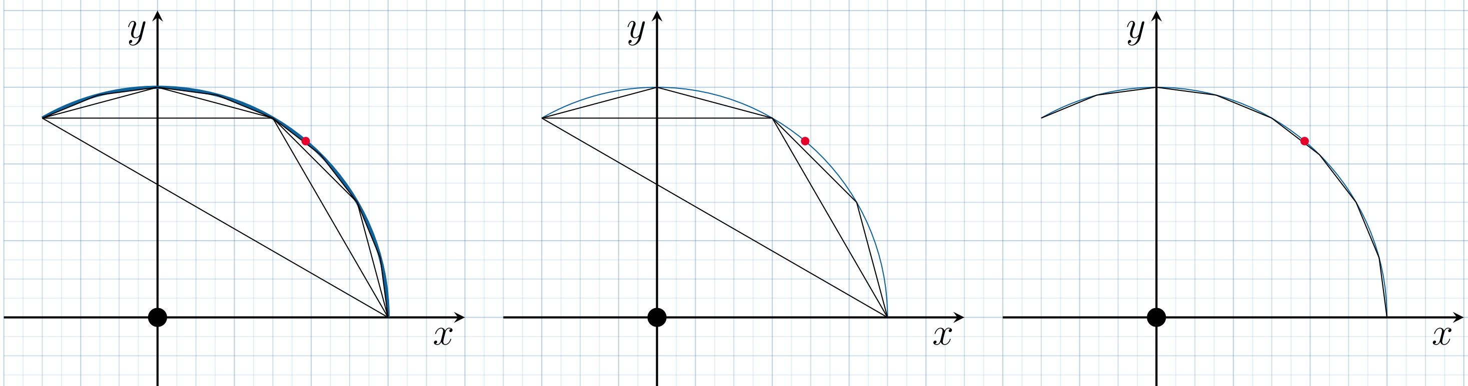
Exercise 4

Given any point p_0 in $\mathscr{C}$, there is a sequence (p_m) of points in $\mathscr{C}$ that lie clockwise from p_0 and a sequence (P_m) of points in $\mathscr{C}$ that lie counterclockwise from p_0, so that (p_m) is counterclockwise ordered and (P_m) is clockwise ordered, and $(\|P_m - p_m\|)$ is a null sequence.

For the prompts below, take p_0 to be a point in Quadrant I that does not lie on the coordinate axes.

(a) Can (p_m) and (P_m) be chosen for each m to be in the vertex set of $g_3(m)$, the regular circumscribed polygon with $3 \cdot 2^m$ edges? Explain your answer.

(b) Sketch a picture that illustrates the meaning of this statement.



Given any point p_0 on $\mathscr{C}$, there is a sequence (p_m) of points in $\mathscr{C}$ that lie clockwise from p_0 and a sequence (P_m) of points in $\mathscr{C}$ that lie counterclockwise from p_0, so that (p_m) is counterclockwise ordered and (P_m) is clockwise ordered, and $(\|P_m - p_m\|)$ is a null sequence.

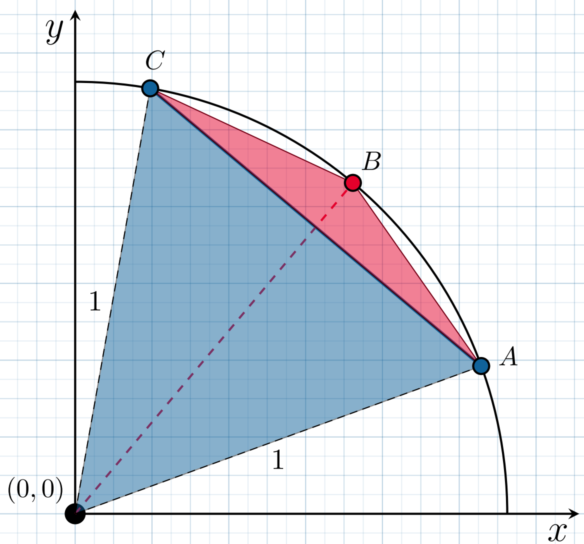
For the prompts below, take p_0 to be a point in Quadrant I that does not lie on the coordinate axes.

(c) Explain how this allows us to define $\mathrm{Frac}(c(t_0))$ for any $c(t_0)$ in Quadrant I, hence for any point in the circle. Your explanation should be just a sketch of a more complete argument.

(d) Will $\mathrm{Frac}(c(t_0))$ necessarily be a rational number?

Denote by $g(n)$ and $G(n)$, respectively, the regular inscribed and regular circumscribed polygons with $3 \cdot 2^n$ edges. Denote by $\Lambda(g(n))$ the perimeter of $g(n)$, $\Lambda(G(n))$ the perimeter of $G(n)$, $\mathcal{A}(g(n))$ the area of $g(n)$, and $\mathcal{A}(G(n))$ the area of $G(n)$.

(a) Explain why the sequences $(\Lambda(g_3(m)))$ and $(\mathcal{A}(g_3(m)))$ are increasing.



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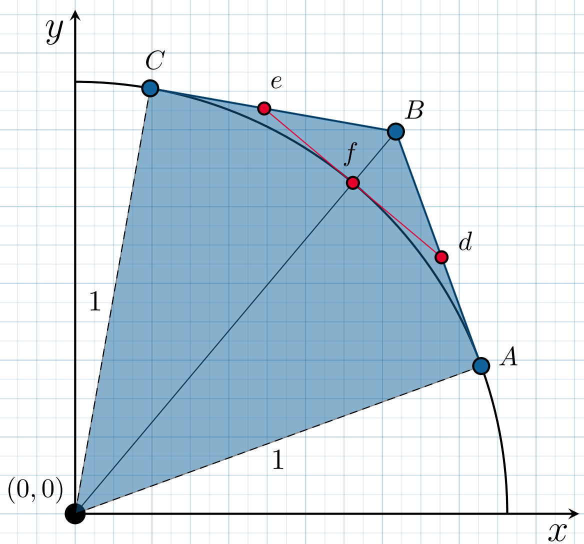
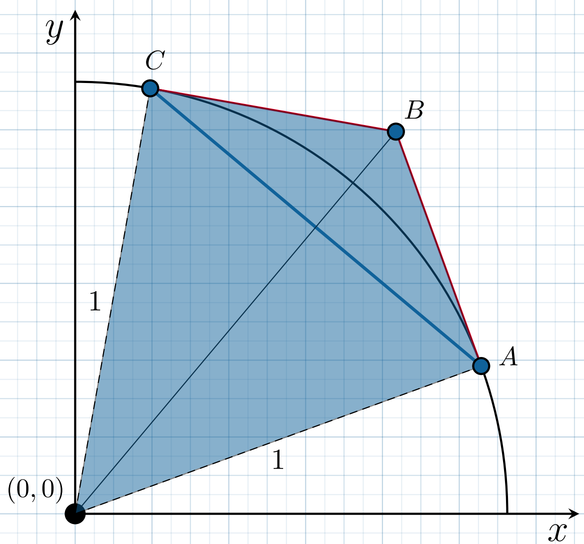
(b) Explain why the sequences $(\Lambda(G_3(m)))$ and $(\mathcal{A}(G_3(m)))$ are decreasing.



Denote by $g(n)$ and $G(n)$, respectively, the regular inscribed and regular circumscribed polygons with $3 \cdot 2^n$ edges. Denote by $\Lambda(g(n))$ the perimeter of $g(n)$, $\Lambda(G(n))$ the perimeter of $G(n)$, $\mathcal{A}(g(n))$ the area of $g(n)$, and $\mathcal{A}(G(n))$ the area of $G(n)$.

(c) Explain why for each m,

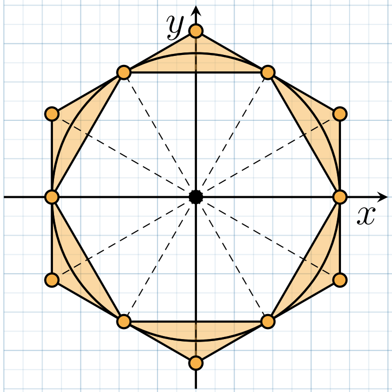
$$\Lambda(g_3(m)) < \Lambda(G_3(m)) \quad \text{and} \quad \mathcal{A}(g_3(m)) < \mathcal{A}(G_3(m)).$$

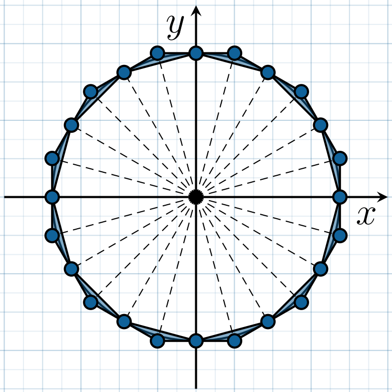


Use the fact that $(\Lambda(G_n) - \Lambda(g_n))$ and $(\mathcal{A}(G_n) - \mathcal{A}(g_n))$ are both null sequences to justify these statements:

(d) There is only one real value that for all n is in the interval $[\Lambda(g_n), \Lambda(G_n)]$.

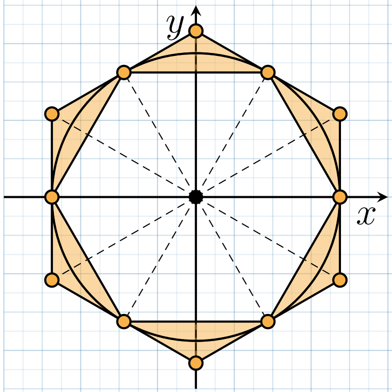
(e) There is only one real value that for all n is in the interval $[\mathcal{A}(g_n), \mathcal{A}(G_n)]$.

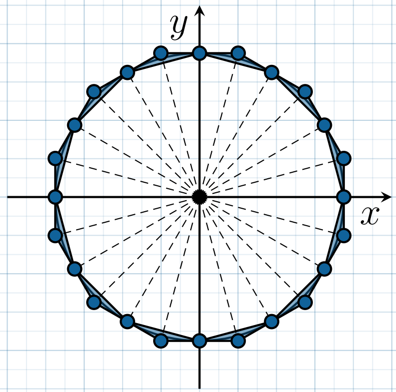




Use the fact that $(\Lambda(G_n) - \Lambda(g_n))$ and $(\mathcal{A}(G_n) - \mathcal{A}(g_n))$ are both null sequences to justify these statements:

(f) The sequences $(\Lambda(g_n))$, $(\mathcal{A}(g_n))$, $(\Lambda(G_n))$, and $(\mathcal{A}(G_n))$ are convergent. To what values do these sequences converge?





(g) Define the circumference of the unit circle to be the unique element given by the value in (d) and the area of the unit circle to be the unique element given by the value in (e). The way in which these quantities are defined appears to depend on the approximation procedure. Does it depend on the procedure?

Exercise 6

The above outlined approach may be modified to directly handle questions about length of arcs of a circle and permits us to define the functions $\sin$ and $\cos$ as functions of a radian angle measure. Following these arguments, it is possible to show that for any null sequence (a_n):

- $(\sin(a_n))$ is a null sequence;
- $(\cos(a_n))$ converges to 1;
- and $\left(\frac{\sin(a_n)}{a_n}\right)$ converges to 1.

Use these facts and the properties of the trigonometric functions to determine for any null sequence (a_n) the following limits:

(a) $\lim\limits_{n\to\infty} \frac{1-\cos(a_n)}{a_n}$;

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Use these facts and the properties of the trigonometric functions to determine for any null sequence (a_n) the following limits:

(b) $\lim\limits_{n\to\infty} \frac{1-\cos(a_n)}{a_n^2}$;

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Use these facts and the properties of the trigonometric functions to determine for any null sequence (a_n) the following limits:

(c) $\lim\limits_{n\to\infty} \frac{\sin(3a_n)}{5a_n}$;

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Use these facts and the properties of the trigonometric functions to determine for any null sequence (a_n) the following limits:

(d) $\lim\limits_{n\to\infty} \frac{\sin(2a_n)}{\sin(7a_n)}$.