Linguistic Mapping

The Principles of Calculus II

V

Finite Approximation

V.2

Sequences

Classroom Exercises

Exercise 1

Precisely respond to these prompts:

(a) For any natural number N, what does it mean for $(a_n)_{n\in\{1,\ldots,N\}}$ to be a finite sequence in a set X? Give an example of such a sequence that takes values in a set other than $\mathbb{R}$.

(b) What does it mean for a $\big($or $(a_n)\big)$ to be a sequence in a set X?

(c) What does it mean for a $\big($or $(a_n)\big)$ to be a null sequence in $\mathbb{R}$?

Exercise 2

It is important to remember that saying that $\mathbb{R}$ has the Archimedean property means that largeness and smallness of real numbers is determined by the natural numbers.

(a) Give three precise and equivalent ways of stating that $\mathbb{R}$ has the Archimedean property.

(b) Use the Archimedean property of $\mathbb{R}$ to show that (a_n) is a null sequence, where (a_n) is given by

$$a_n = \frac{1}{2n+7}.$$

Exercise 3

Utilizing the Archimedean property of the real numbers to demonstrate that a sequence is a null sequence ultimately reduces to making an estimate. Exact calculations are not necessary and are typically not very helpful. The point is to use estimates to simplify your calculations.

The goal of this exercise is to use the Archimedean property of $\mathbb{R}$ to show that (a_n) is a null sequence, where (a_n) is given by

$$a_n = \frac{n+1}{\sqrt{n^3 - n + 2}}.$$

(a) For each n, is the numerator of n smaller than $2n$?

The goal of this exercise is to use the Archimedean property of $\mathbb{R}$ to show that (a_n) is a null sequence, where (a_n) is given by

$$a_n = \frac{n+1}{\sqrt{n^3 - n + 2}}.$$

(b) Show that for each n,

$$\frac{1}{4}n^3 < n^3 - n + 2.$$

(c) Use (a) and (b) to find a convenient upper bound for a_n.

The goal of this exercise is to use the Archimedean property of $\mathbb{R}$ to show that (a_n) is a null sequence, where (a_n) is given by

$$a_n = \frac{n+1}{\sqrt{n^3 - n + 2}}.$$

(d) Now use the Archimedean property of $\mathbb{R}$ to show that (a_n) is a null sequence.

(e) Would it have been possible to come up with a tighter estimate? How important would that be for our current goal of demonstrating that (a_n) is a null sequence?

Exercise 4

Null sequences may be though of as sequences of errors and can make precise the idea of a sequence converging to a nonzero limit.

(a) Directly use the concept of a null sequence to define what it means for a sequence to converge to a limit L.

(b) Show that the sequence (a_n) converges to 5, where (a_n) is given by

$$a_n = \frac{5n^2 + 2n - 1}{n^2 - n + 7}.$$

Exercise 5

Convergent sequences have important boundedness properties.

(a) State what it means for a sequence to be bounded.

(b) Give an example of a sequence that has infinitely many elements in its range, that is bounded, but that is not convergent.

(c) Give an example of a sequence that is unbounded but bounded below.

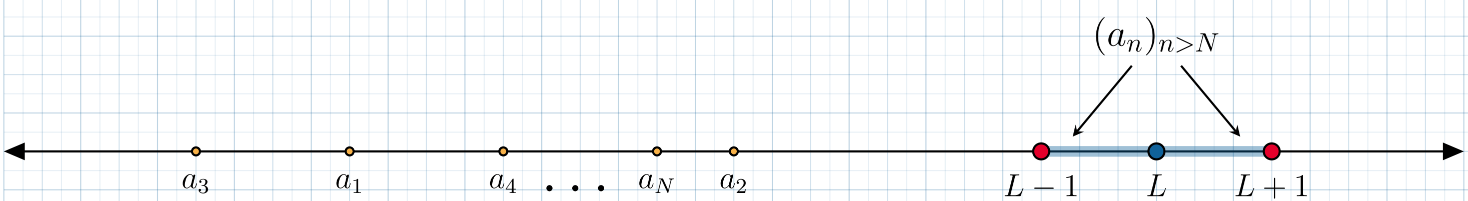
(d) Give an example of a sequence that is unbounded but bounded above.

(e) Give an example of a sequence that is unbounded both above and below.

Exercise 6

Convergent sequences have important boundedness properties. In this exercise, we will show that if (a_n) is convergent, then (a_n) is bounded.

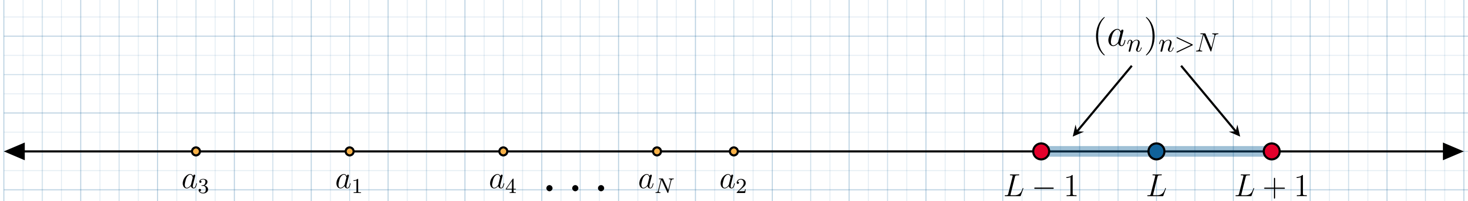
(a) It is helpful to think heuristically about convergent sequences and to develop a language that facilitates this thinking. With this in mind, explain in plain English the meaning of the *head* and *tail* of a convergent sequence.



Convergent sequences have important boundedness properties. In this exercise, we will show that if (a_n) is convergent, then (a_n) is bounded.

(b) The picture below reflects the intuition about the proof of the statement. Explain the meaning of the picture in plain English using the idea of a head and a tail of a sequence.

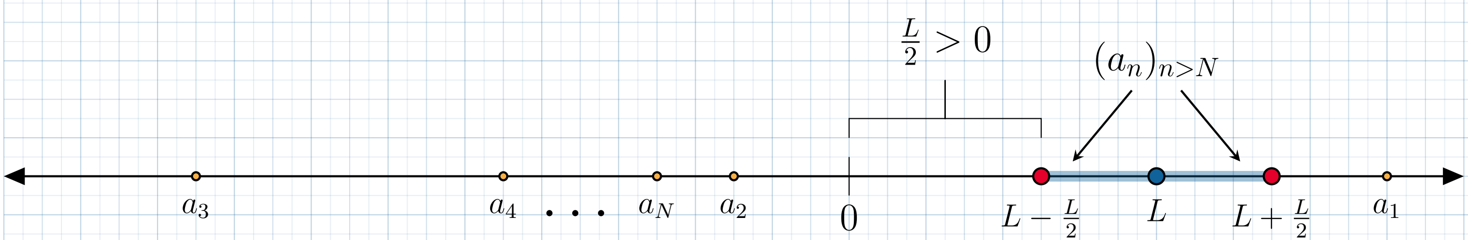
(c) Use the intuition from the picture to write a formal argument.



Convergent sequences have important boundedness properties. In this exercise, we will show that if (a_n) never takes on the value 0 and a_n converges to a real number L that is non-zero, then $\left(\frac{1}{a_n}\right)$ is bounded.

(a) The picture below reflects the intuition about the proof of the statement. Explain the meaning of the picture in plain English using the idea of a head and a tail of a sequence.

(b) Use the intuition from the picture to write a formal argument.



Exercise 8

Precisely express these statements about null sequences:

(a) (Sandwich theorem for null sequences)

Any sequence that is bounded above by one null sequence and bounded below by another null sequence is itself a null sequence;

(b) (Limit law for sums of null sequences)

Any sum of two null sequences is a null sequence;

(c) (Limit law for products of null sequences)

The product of any two null sequences is a null sequence;

(d) (Limit law for bounded and null sequence products)

The product of any bounded sequence with any null sequence is a null sequence.

Use the known facts about null sequences to prove these more general statements.
Take (a_n), (b_n), and (c_n) to be any sequences in $\mathbb{R}$.

(a) (Sandwich theorem)

If (a_n) converges to L, (c_n) converges to L, and for any n,

$$a_n \leq b_n \leq c_n,$$

then (b_n) converges to L.

Use the known facts about null sequences to prove these more general statements.
Take (a_n), (b_n), and (c_n) to be any sequences in $\mathbb{R}$.

(b) (Limit law for sums)

If (a_n) converges to L and (b_n) converges to M, then $(a_n + b_n)$ converges to $L + M$.

Use the known facts about null sequences to prove these more general statements. Take (a_n), (b_n), and (c_n) to be any sequences in $\mathbb{R}$.

(c) (Limit law for products)

If (a_n) converges to L and (b_n) converges to M, then $(a_n b_n)$ converges to LM.

Use the known facts about null sequences to prove these more general statements.
Take (a_n), (b_n), and (c_n) to be any sequences in $\mathbb{R}$.

(d) (Limit law for quotients)

If (a_n) converges to L, (b_n) converges to M, b_n is never 0, and M is non-zero, then $\left(\frac{a_n}{b_n}\right)$ converges to $\frac{L}{M}$.

Exercise 10

Use the fact that there is a positive sequence (ε_n) so that for each n,

$$2^{\frac{1}{n}} = 1 + \varepsilon_n,$$

to show that $(2^{\frac{1}{n}})$ is convergent and to determine its limit.

Exercise 11

For each of these choices of sequence (a_n), use the sandwich theorem to determine the quantity

$$\lim_{n\to\infty} a_n.$$

(a) $a_n = (2^n + 7^n)^{\frac{1}{n}}$;

(b) $a_n = \frac{5n^2+n\cos(n)}{3n^2+4}$.

Exercise 12

Take (a_n), (b_n), and (c_n) to be sequences with

$$\lim_{n\to\infty} a_n = 4, \quad \lim_{n\to\infty} b_n = 2 \quad \text{and} \quad \lim_{n\to\infty} c_n = 5.$$

Use the limit laws to evaluate these limits:

(a) $\lim_{n\to\infty} a_n b_n c_n$;

(b) $\lim_{n\to\infty} a_n^2 - 3b_n$;

(c) $\lim_{n\to\infty} \frac{a_n - c_n}{b_n}$;

(d) $\lim_{n\to\infty} \frac{6n^2+n+2}{4n^2-n+7}$.

Exercise 13

Use the limit laws to evaluate $\lim_{n\to\infty} a_n$, for each of these choices of sequence (a_n):

(a) $a_n = \sqrt{4 + \frac{1}{n}}$;

(b) $a_n = n\left(\sqrt{4 + \frac{1}{n}} - 2\right)$.

Exercise 14

The nested interval property guarantees the existence of certain limits of sequences in $\mathbb{R}$.

(a) Precisely state what it means to say that $\mathbb{R}$ has the nested interval property.

(b) Take (a_n) to be an increasing sequence and (c_n) to be a decreasing sequence, both in $\mathbb{R}$. For any sequence (b_n) in $\mathbb{R}$, if for all n,

$$a_n \leq b_n \leq c_n,$$

and if $(c_n - a_n)$ is a null sequence, then show that (b_n) is convergent in $\mathbb{R}$. Hint: You will need to use the nested interval property of $\mathbb{R}$.

(c) The above theorem is not true if $\mathbb{R}$ is everywhere replaced by $\mathbb{Q}$. Can you show that it is not true? Hint: $\sqrt{2}$ is not a rational number.

Exercise 15

Monotone sequence are very important in our subject because of a certain existence theorem about the limits of bounded monotone sequences.

(a) What does it mean for a sequence to be monotone?

(b) Show that (a_n) is increasing and (b_n) is decreasing, where

$$a_n = \tfrac{n}{n+1} \quad \text{and} \quad b_n = \tfrac{n+2}{n+1}.$$

(c) State the monotone convergence theorem for sequences. Why does part (b) of the previous question not automatically give us a proof of this theorem? How might you prove this theorem?

Exercise 16

Take (a_n) to be the sequence that is given recursively by

$$\begin{cases} a_1 = 3 \\ a_{n+1} = \frac{1}{2}\left(a_n + \frac{3}{a_n}\right). \end{cases}$$

(a) Explain in plain English what it means for a sequence to be recursively defined.

(b) Show that (a_n) is decreasing and bounded.

(c) Justify that (a_n) converges and determine its limit.

Exercise 17

Prior to working through the examples below, state what it means for a sequence to diverge to infinity and what it means for a sequence to diverge negative infinity.

For each of these choices of sequence (a_n), use the Archimedean property of $\mathbb{R}$ to show that (a_n) diverges to either infinity or negative infinity:

(a) $a_n = 3^n$;

(b) $a_n = n^{\frac{1}{5}}$;

(c) $a_n = -n^2$.

Exercise 18

For each of these choices of sequence (a_n), determine whether (a_n) is divergent and whether (a_n) diverges to ∞ or $-\infty$:

(a) (n^3);

(b) $\left((-1)^n + \frac{1}{n}\right)$.