Linguistic Mapping

The Principles of Calculus I

V

Finite Approximation

V.1

The Elementary Notion of Area

Classroom Exercises

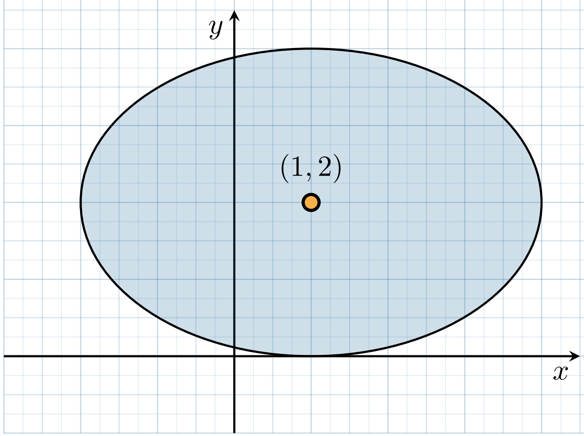
Exercise 1

The notion of curve, closed curve, and of a point being inside or outside the region bounded by a closed curve all make sense at this point at an intuitive level. However, making these ideas precise poses a major challenge that our course of study must address.

In some cases, these ideas may already be made precise. As an example, take E to be the solid ellipse that is given by the equation

$$\frac{(x-1)^2}{9} + \frac{(y-2)^2}{4} \leq 1.$$

(a) Explain in plain English the meaning of ∂E and identify a formula for ∂E.



As an example, take E to be the solid ellipse that is given by the equation

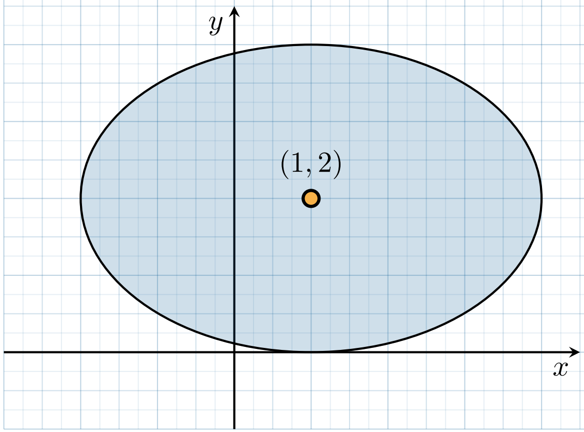
$$\frac{(x-1)^2}{9} + \frac{(y-2)^2}{4} \leq 1.$$

(b) Identify a formula for a counterclockwise oriented path c_+ that parameterizes ∂E.

(c) Simulate the motion of a particle whose position at time t is $c_+(t)$.

(d) Identify a formula for a clockwise oriented path c_- that parameterizes ∂E.

(e) Simulate the motion of a particle whose position at time t is $c_-(t)$.

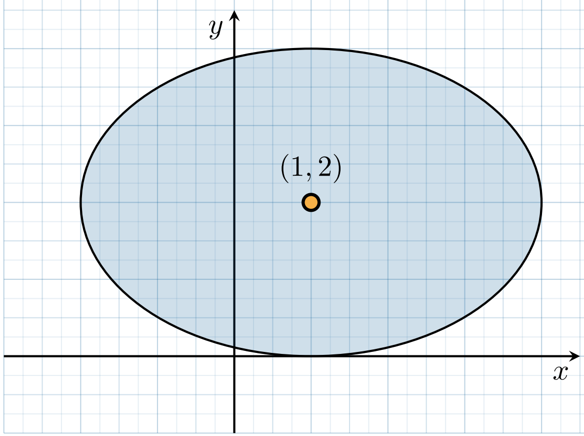


As an example, take E to be the solid ellipse that is given by the equation

$$\frac{(x-1)^2}{9} + \frac{(y-2)^2}{4} \leq 1.$$

(f) Explain in plain English what it means for a point p to be inside ∂E.

(g) Show that the point $(2, 3)$ is inside ∂E.

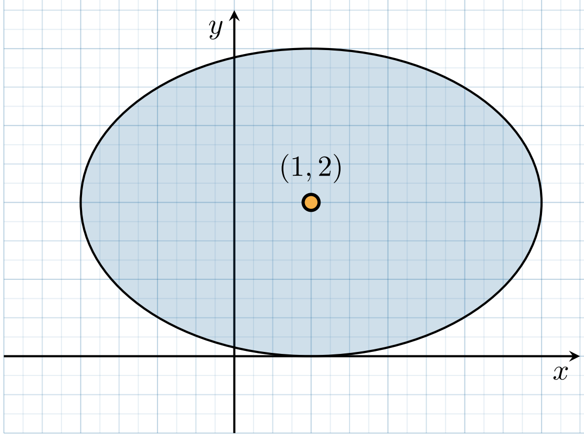


As an example, take E to be the solid ellipse that is given by the equation

$$\frac{(x-1)^2}{9} + \frac{(y-2)^2}{4} \leq 1.$$

(h) Explain in plain English what it means for a point p to be outside ∂E.

(i) Show that the point $(3, 4)$ is outside ∂E.

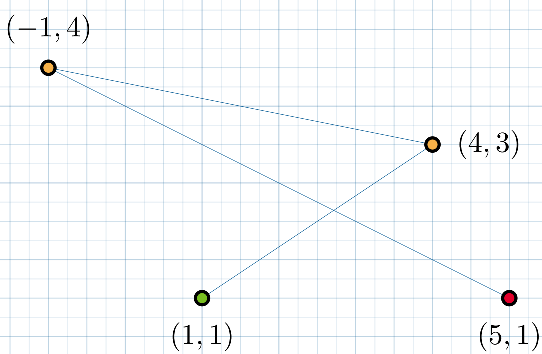


We currently lack the language to work in a general setting, making it necessary to restrict to working only with piecewise linear functions.

Take c to be a continuous, piecewise linear path with domain equal to $[0, 5]$, epoch $e(c)$, and ordered vertex set $V(c)$ respectively given by

$$e(c) = (0, 1, 2, 5) \quad \text{and} \quad V(c) = \big((1, 1), (4, 3), (-1, 4), (5, 1)\big).$$

(a) Explain in plain English what it means for c to satisfy these conditions.

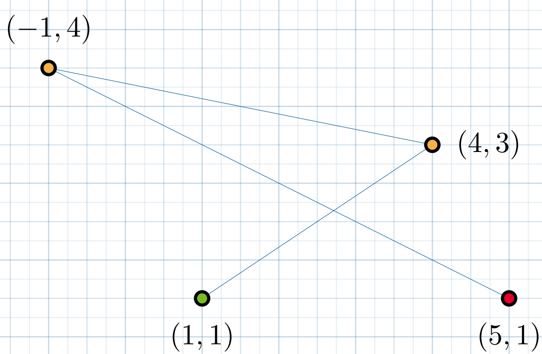


Take c to be a continuous, piecewise linear path with domain equal to $[0, 5]$, epoch $e(c)$, and ordered vertex set $V(c)$ respectively given by

$$e(c) = (0, 1, 2, 5) \quad \text{and} \quad V(c) = \big((1, 1), (4, 3), (-1, 4), (5, 1)\big).$$

(b) Identify an equation for c.

(c) Simulate the motion of a particle whose position at time t is given by $c(t)$, where t is in $[0, 5]$.

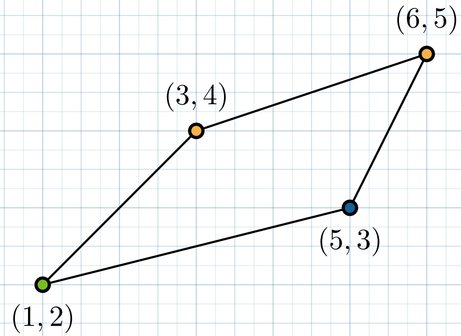


Exercise 3

Explain in plain English what is meant by a polygonal path c with an ordered vertex set $\big((1,2),(5,3),(6,5),(3,4)\big)$ and epoch $(1,2,5,9,11)$.

(a) Identify a formula for c.

(b) Explain in plain English the relationship between the words "polygonal path" and "polygonal curve".

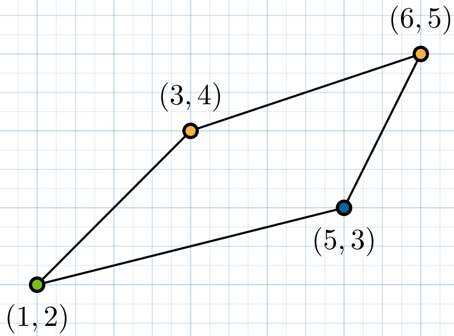


Explain in plain English what is meant by a polygonal path c with an ordered vertex set $\big((1,2),(5,3),(6,5),(3,4)\big)$ and epoch $(1,2,5,9,11)$.

(c) Use c to identify the boundary $\partial\square\big((1,2),(5,3),(6,5),(3,4)\big)$.

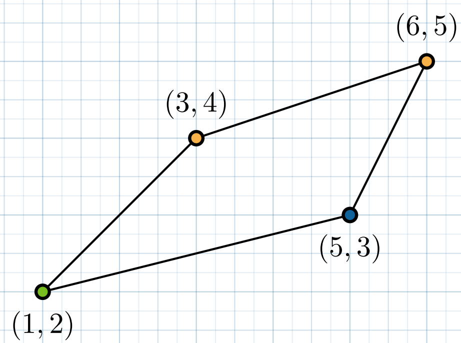
(d) Describe $\partial\square\big((1,2),(5,3),(6,5),(3,4)\big)$ as a union of line segments.

(e) Identify the oriented boundary $\partial_o\square\big((1,2),(5,3),(6,5),(3,4)\big)$.



Explain in plain English what is meant by a polygonal path c with an ordered vertex set $\big((1,2),(5,3),(6,5),(3,4)\big)$ and epoch $(1,2,5,9,11)$.

(f) Explain in plain English what is meant for a point to lie inside the polygon $\square\big((1,2),(5,3),(6,5),(3,4)\big)$.



Exercise 4

Take $\square(p_1, \ldots, p_n)$ to be a polygon. Explain in plain English the difference between the vertex set $\{p_1, p_2, \ldots, p_n\}$, the ordered vertex set $(p_1, p_2, \ldots, p_n)$, the boundary $\partial\square(p_1, p_2, \ldots, p_n)$, and the oriented boundary $\partial_o\square(p_1, p_2, \ldots, p_n)$.

Exercise 5

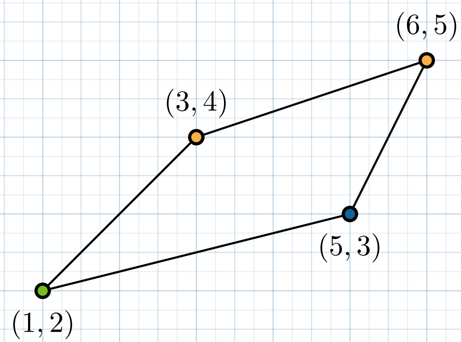
The polygon $\square\big((1,2),(5,3),(6,5),(3,4)\big)$ is sketched below. Identify the meaning of these symbols:

(a) $\{(1,2),(5,3),(6,5),(3,4)\}$,

(b) $\big((1,2),(5,3),(6,5),(3,4)\big)$,

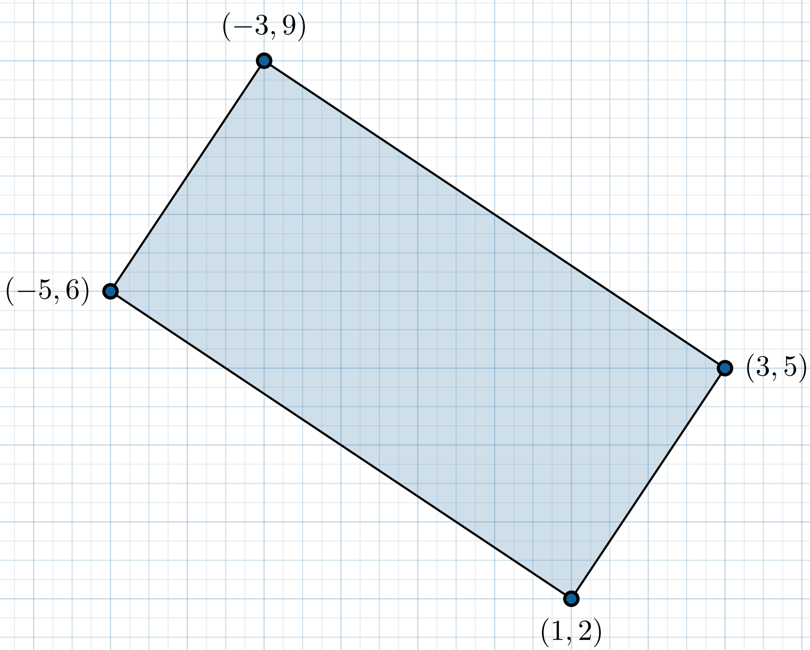
(c) $\left\{\overline{(1,2)(5,3)},\overline{(5,3)(6,5)},\overline{(6,5)(3,4)},\overline{(3,4)(1,2)}\right\}$,

(d) $\left\{\overrightarrow{(1,2)(5,3)},\overrightarrow{(5,3)(6,5)},\overrightarrow{(6,5)(3,4)},\overrightarrow{(3,4)(1,2)}\right\}$.



Exercise 6

Take R to be the rectangle with vertices $(1, 2)$, $(3, 5)$, $(-3, 9)$, and $(-5, 6)$. Determine the area that is bounded by ∂R.



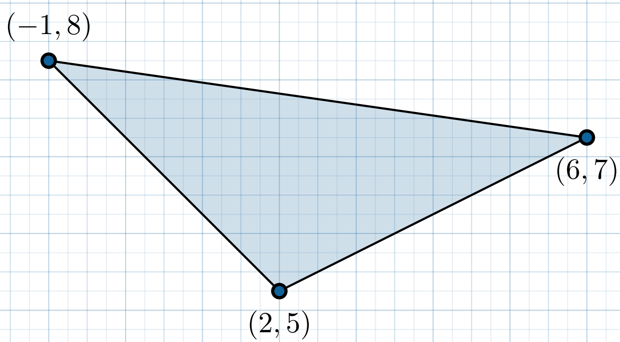
Exercise 7

The solid triangle $\Delta(2,5)(-1,8)(6,7)$ is sketched below.

(a) Express $\Delta(2,5)(-1,8)(6,7)$ as union of disjoint line segments.

(b) Express $\Delta(2,5)(-1,8)(6,7)$ as a union of line segments that all share a common vertex. To improve typography, denote by R this triangle.

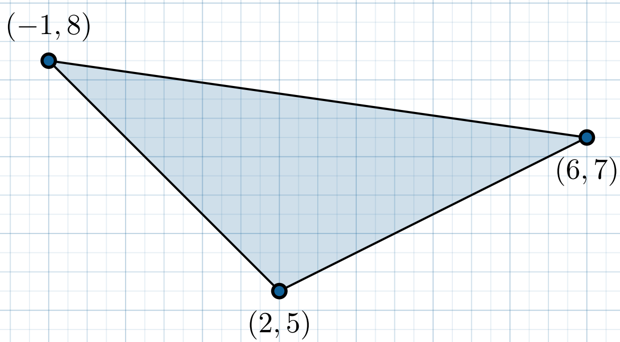
(c) Simulate both (a) and (b) by varying the line segments by a single parameter.



The solid triangle $\Delta(2,5)(-1,8)(6,7)$ (denoted R) is sketched below.

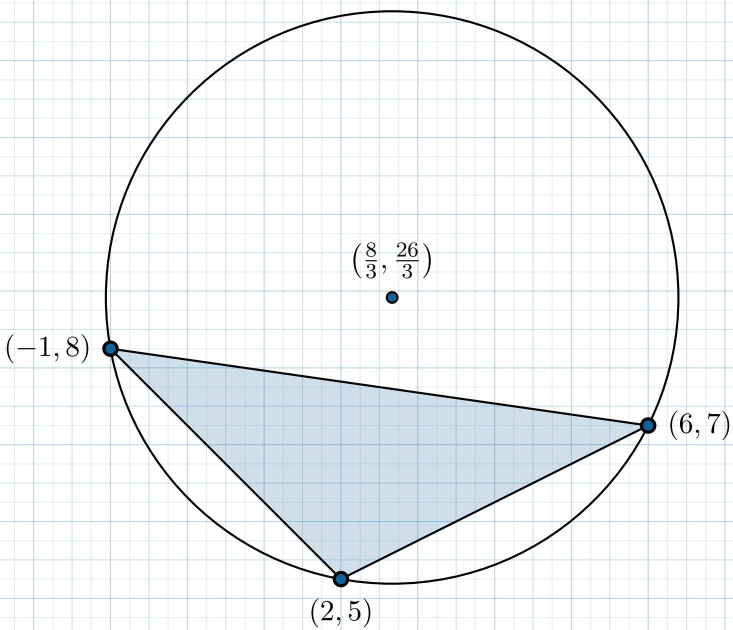
(d) Determine whether the point $(3,6)$ is inside R or outside R.

(e) Determine whether the point $(3,8)$ is inside R or outside R.



The solid triangle $\Delta(2,5)(-1,8)(6,7)$ is sketched below.

(f) Identify the circumcircle of R, and use the circumcircle to determine whether R is positively or negatively oriented.

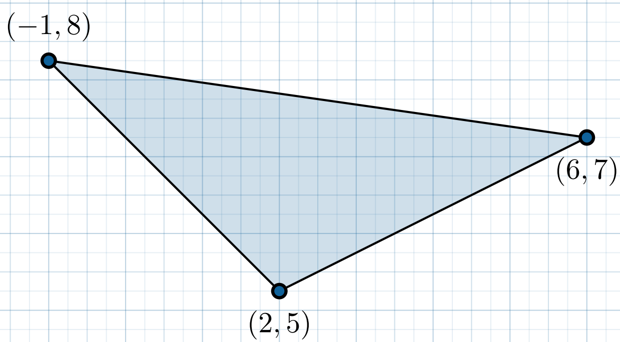


The solid triangle $\Delta(2,5)(-1,8)(6,7)$ is sketched below.

(g) Parameterize the boundary of R with a unit speed polygonal path.

(h) Identify the boundary ∂R of R.

(i) Identify the oriented boundary $\partial_o R$ of R.



Exercise 8

For any point p in $\mathbb{R}^2$, the coordinates of p in the standard (x, y)-coordinate plane are given by $(x(p), y(p))$. The Shoelace Formula for the Triangle, states that if $\Delta p_1 p_2 p_3$ is positively oriented, then

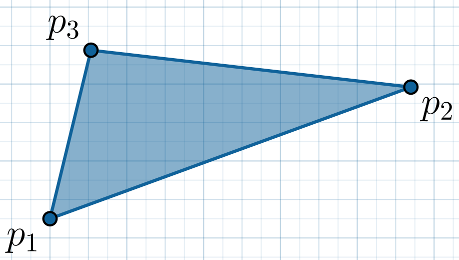
$$\begin{aligned}\mathcal{A}(\Delta p_1 p_2 p_3) = \tfrac{1}{2}\,&(x(p_1)y(p_2) + x(p_2)y(p_3) + x(p_3)y(p_1) \\ &-y(p_1)x(p_2) - y(p_2)x(p_3) - y(p_3)x(p_1))\,.\end{aligned}$$

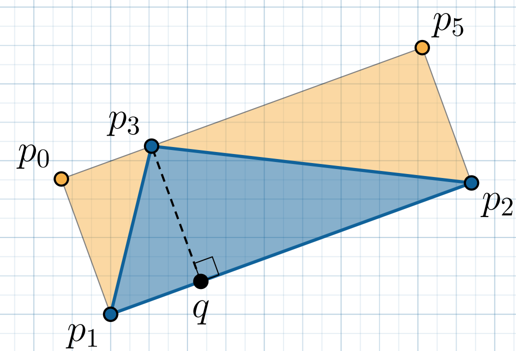
Furthermore, if $\Delta p_1 p_2 p_3$ is negatively oriented, then

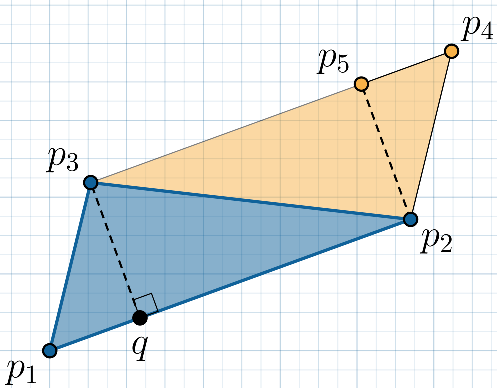
$$\begin{aligned}\mathcal{A}(\Delta p_1 p_2 p_3) = -\tfrac{1}{2}\,&(x(p_1)y(p_2) + x(p_2)y(p_3) + x(p_3)y(p_1) \\ &-y(p_1)x(p_2) - y(p_2)x(p_3) - y(p_3)x(p_1))\,.\end{aligned}$$

The goal of this exercise is to calculate the area, $\mathcal{A}(\Delta p_1 p_2 p_3)$, of the triangle $\Delta p_1 p_2 p_3$ in terms of the coordinates of p_1, p_2, and p_3.

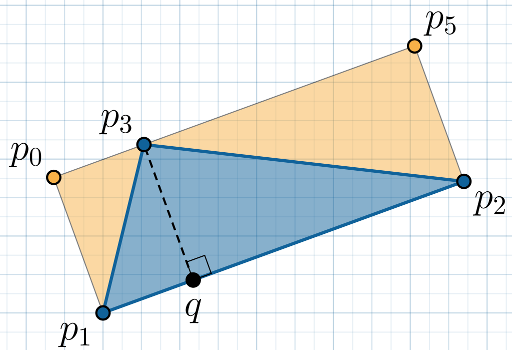
If $\Delta p_1p_2p_3$ can, up to some rotation, be sketched as below, then identify the vector $p_3 - q$ to determine the area of $\Delta p_1p_2p_3$.

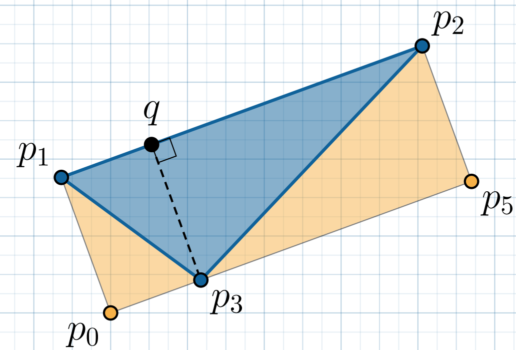






Where did the fact that the triangle is positively oriented come into the derivation?





Exercise 9

For any oriented line segment $\overrightarrow{p_1p_2}$ denote $\alpha\left(\overrightarrow{p_1p_2}\right)$, the surveyor's symbol for the oriented line segment $\overrightarrow{p_1p_2}$, by the quantity

$$\alpha\left(\overrightarrow{p_1p_2}\right) = x(p_1)y(p_2) - x(p_2)y(p_1).$$

(a) Use the surveyor's symbol to express the signed area of $\Delta(2,5)(-1,8)(6,7)$ in terms of the elements of $\partial_o\Delta(2,5)(-1,8)(6,7)$.

(b) Determine the area of the triangle $\Delta(2,5)(-1,8)(6,7)$.

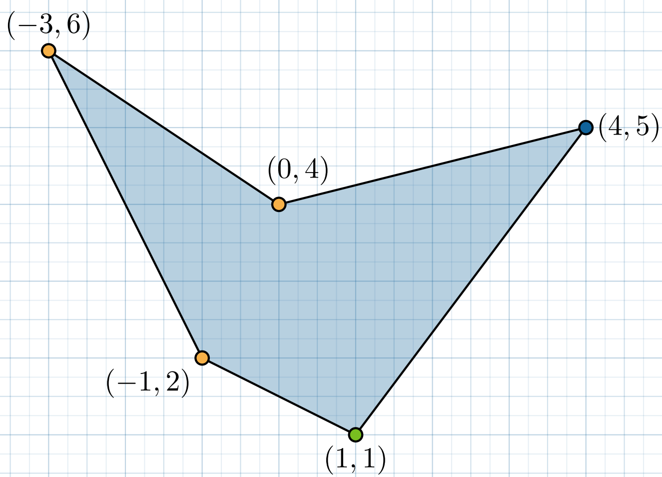
(c) Determine whether the triangle is positively or negatively oriented.

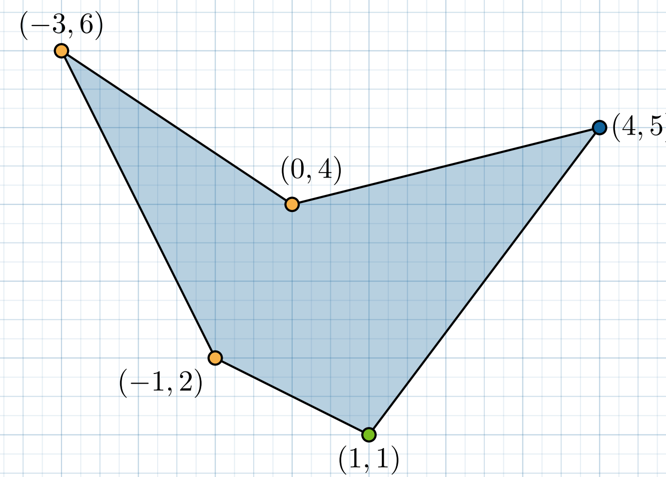
Exercise 10

Take $\big((1,1),(4,5),(0,4),(-3,6),(-1,2)\big)$ to be an ordered vertex set of a polygon P.

(a) Sketch two different triangulations for P, and be sure that one includes a vertex that is inside P.

(b) Calculate the area of each triangle in each of the triangulations using the surveyor's symbol and determine the area of P.





Take $\big((1,1),(4,5),(0,4),(-3,6),(-1,2)\big)$ to be an ordered vertex set of a polygon P.

(c) Will the calculation of the area depend on the triangulation? If so, explain why. If not, explain why not.

(d) Rewrite the formula for the area in terms of the surveyor symbol for the elements of $\partial_o P$.