

*Linguistic Mapping*

# The Principles of Calculus I

IV

Symmetry

IV.3

Symmetric Change

*Classroom Exercises*

Copyright © 2024 by Bryan Carrillo and David Weisbart.

All rights reserved. No part of this publication may be reproduced or transmitted in any form or by any means, electronic or mechanical, including photocopy, recording, or any information storage and retrieval system, without permission in writing from Bryan Carrillo and David Weisbart.

## Exercise 1

Two symmetries for the change that a quantity experiences are restrictive enough to uniquely determine the functions:

On any two intervals of the same length, the quantity changes by either

- the same **amount** (a linear model of change), or
- the same factor (an exponential model of change).

(a) Take  $f$  to be a function that changes linearly with respect to  $t$ , so that  $f(t)$  is the numerical value of the quantity at  $t$ ,  $M$  is the **amount** by which  $f$  changes between 0 and 1, and  $f(0)$  is the numerical value of  $f$  at time 0. Identify a formula for  $f(1)$ .

Two symmetries for the change that a quantity experiences are restrictive enough to uniquely determine the functions:

On any two intervals of the same length, the quantity changes by either

- the same **amount** (a linear model of change), or
- the same factor (an exponential model of change).

(b) Take  $a$  to be the amount by which  $f$  changes on  $[0, \frac{1}{n}]$ . Identify a formula for  $f(\frac{1}{n})$ .

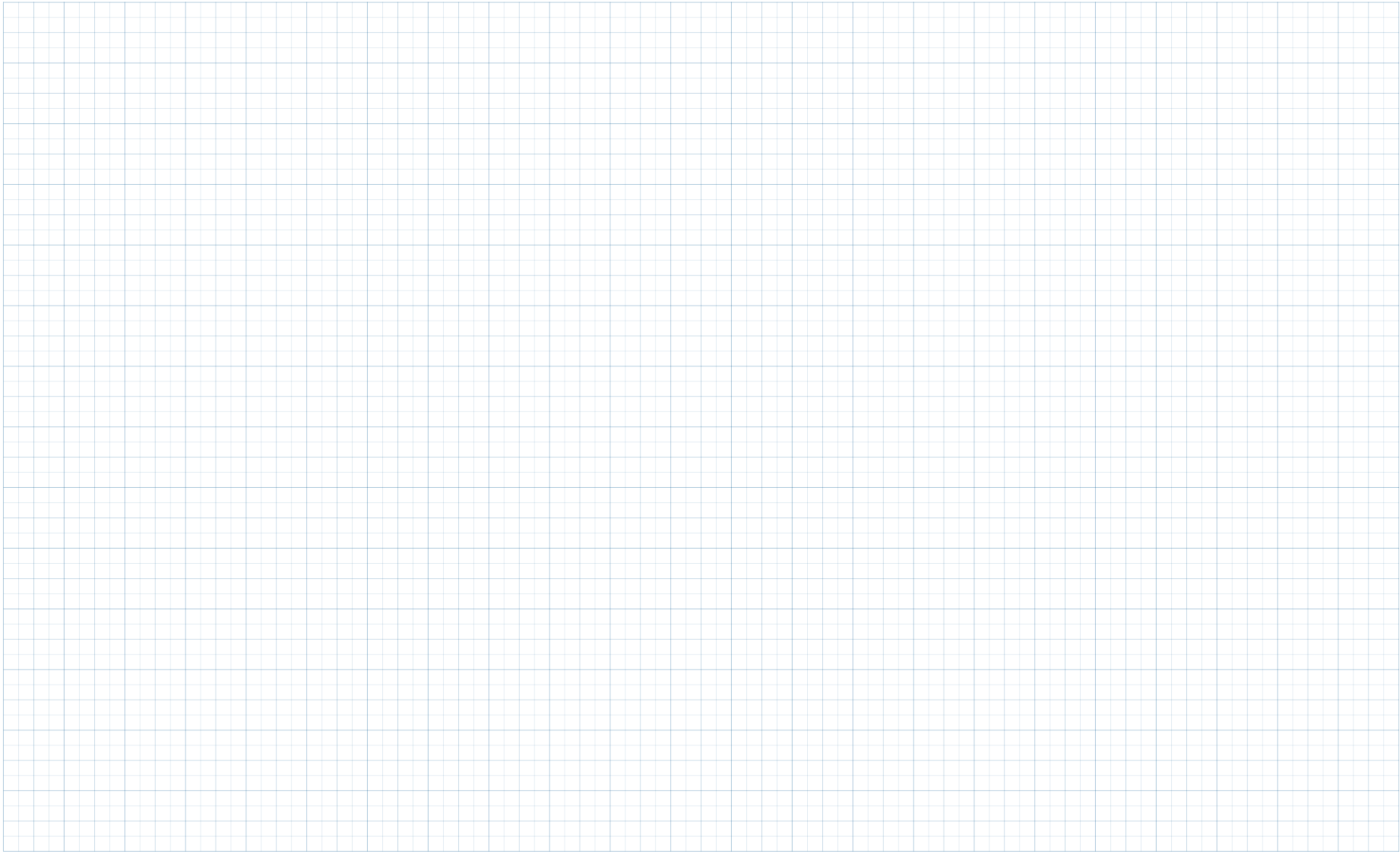
Two symmetries for the change that a quantity experiences are restrictive enough to uniquely determine the functions:

On any two intervals of the same length, the quantity changes by either

- the same **amount** (a linear model of change), or
- the same factor (an exponential model of change).

(c) How does  $f$  change on each of the intervals  $[0, \frac{1}{n}]$ ,  $[\frac{1}{n}, \frac{2}{n}]$ ,  $[\frac{2}{n}, \frac{3}{n}]$ , and so on?

(d) Determine  $a$  in terms of  $M$  and  $n$ .



Two symmetries for the change that a quantity experiences are restrictive enough to uniquely determine the functions:

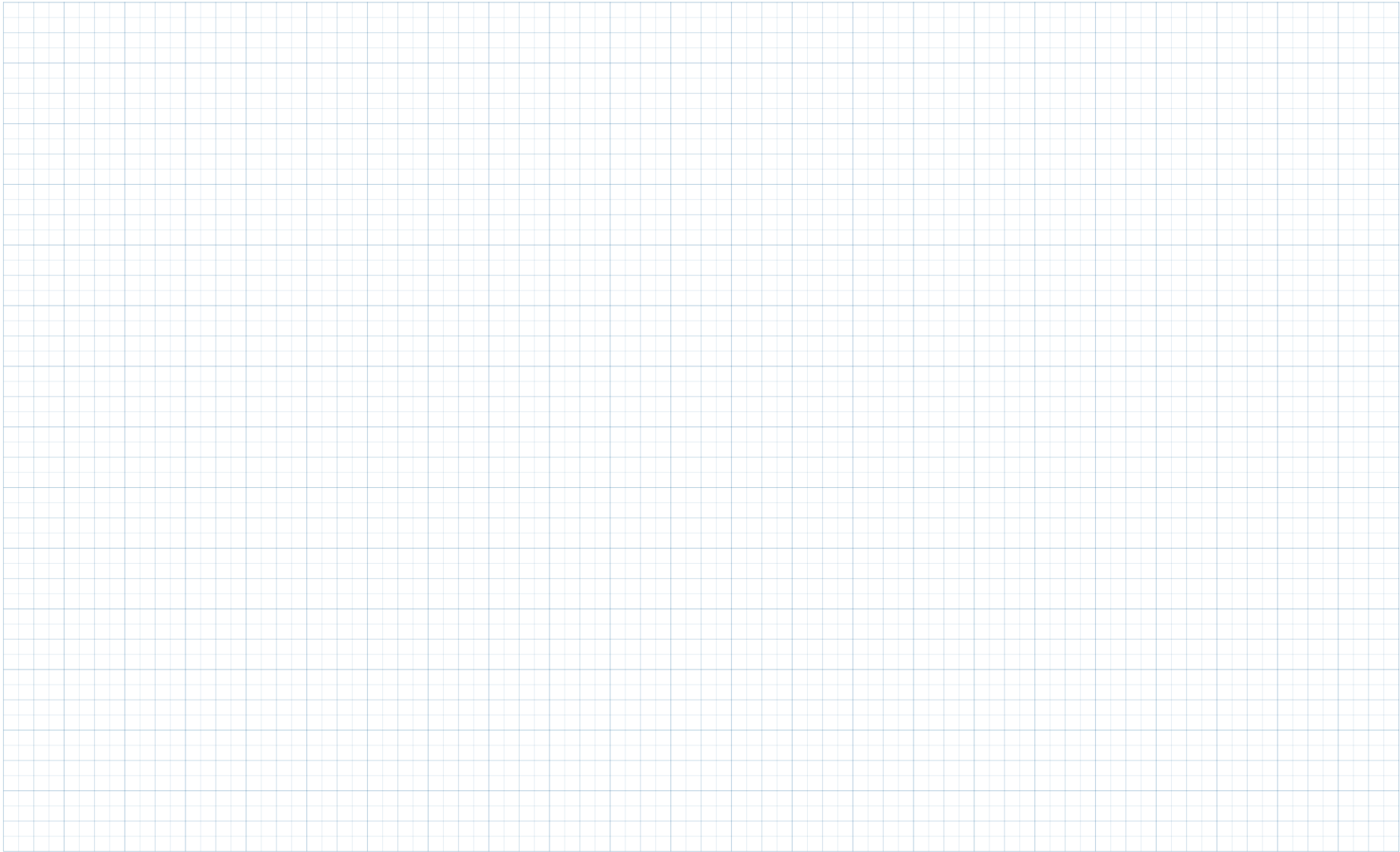
On any two intervals of the same length, the quantity changes by either

- the same **amount** (a linear model of change), or
- the same factor (an exponential model of change).

(e) For any fraction  $\frac{m}{n}$ , determine  $f\left(\frac{m}{n}\right)$ .

(f) For any rational number  $t$ , determine  $f(t)$ .

Note: If  $f$  is assumed to be “nice enough”, then for any real number  $t$ , the formula for  $f(t)$  is the same as in the case when  $t$  is any rational number.



## Exercise 2

The following data set is associated with a quantity that experiences linear change.

- (a) Express the associated model as a function of time.
- (b) Identify the minimal number of data points needed in this table to determine the model.

$t$	$A(t)$
0	4
1	8
2	12
3	16



### Exercise 3

The following data set is associated with a quantity that experiences linear change. Express the associated model as a function of time.

$t$	$A(t)$
3	4
6	8
9	12
12	16

## Exercise 4

Two symmetries for the change that a quantity experiences are restrictive enough to uniquely determine the functions:

On any two intervals of the same length, the quantity changes by either

- the same amount (a linear model of change), or
- the same **factor** (an exponential model of change).

(a) Take  $f$  to be a function that changes exponentially with respect to  $t$ , so that  $f(t)$  is the

numerical value of the quantity at  $t$ ,  $M$  is the **factor** by which  $f$  changes between 0 and 1, and  $f(0)$  is the numerical value of  $f$  at time 0. Identify a formula for  $f(1)$ .

Two symmetries for the change that a quantity experiences are restrictive enough to uniquely determine the functions:

On any two intervals of the same length, the quantity changes by either

- the same amount (a linear model of change), or
- the same **factor** (an exponential model of change).

(b) Take  $a$  to be the **factor** by which  $f$  changes on  $[0, \frac{1}{n}]$ . Identify a formula for  $f(\frac{1}{n})$ .

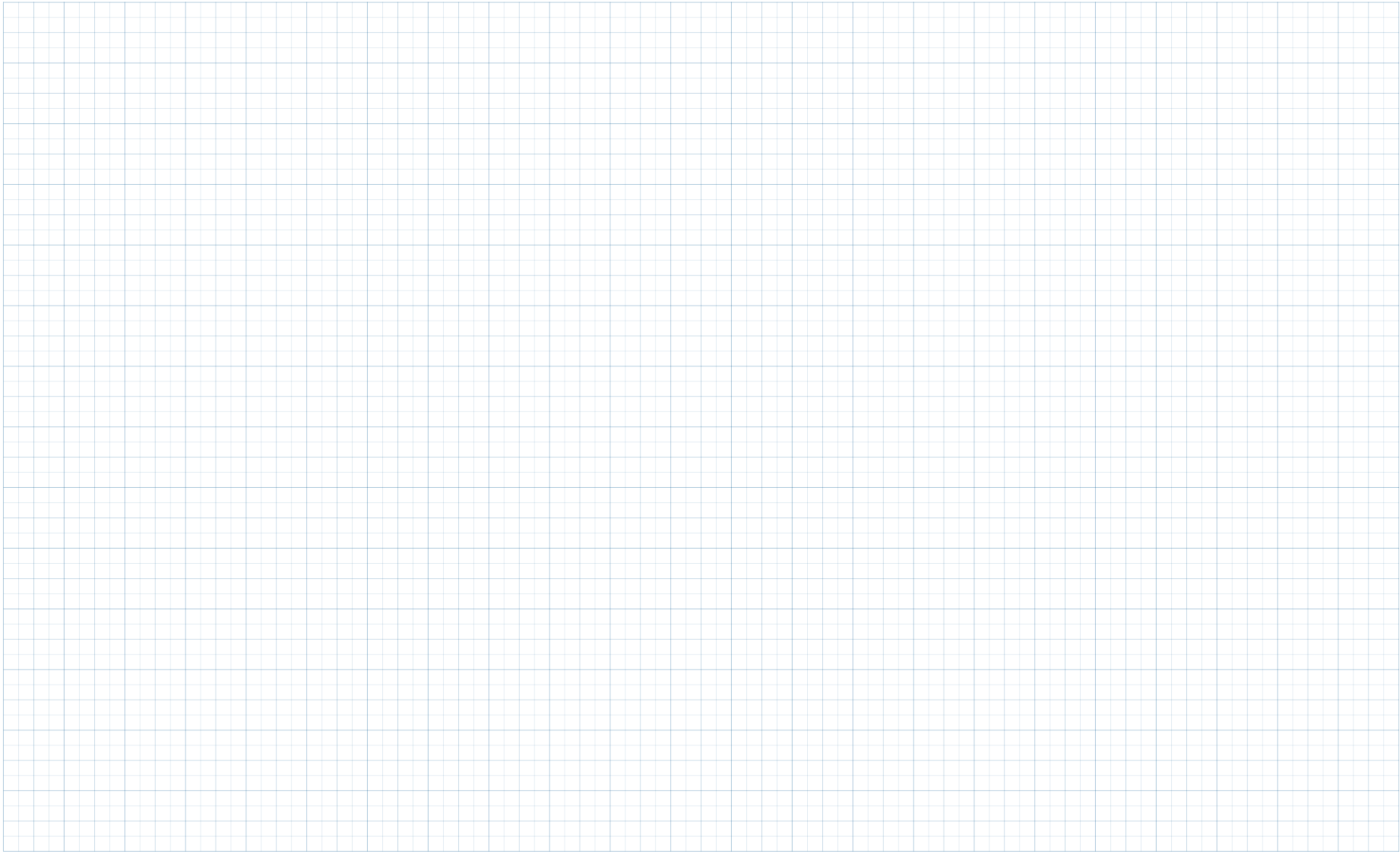
Two symmetries for the change that a quantity experiences are restrictive enough to uniquely determine the functions:

On any two intervals of the same length, the quantity changes by either

- the same amount (a linear model of change), or
- the same **factor** (an exponential model of change).

(c) How does  $f$  change on each of the intervals  $[0, \frac{1}{n}]$ ,  $[\frac{1}{n}, \frac{2}{n}]$ ,  $[\frac{2}{n}, \frac{3}{n}]$ , and so on?

(d) Determine  $a$  in terms of  $M$  and  $n$ .



Two symmetries for the change that a quantity experiences are restrictive enough to uniquely determine the functions:

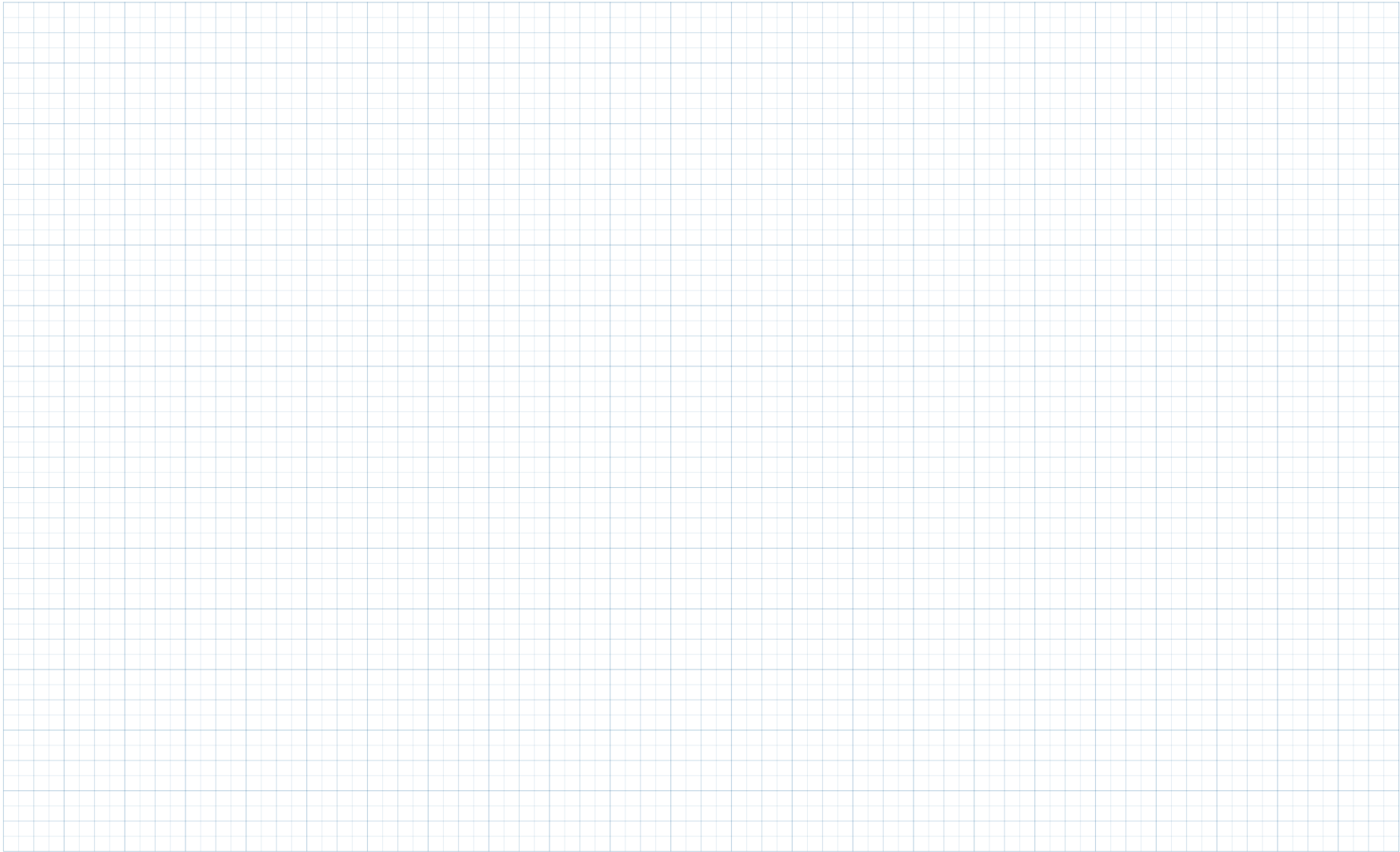
On any two intervals of the same length, the quantity changes by either

- the same amount (a linear model of change), or
- the same **factor** (an exponential model of change).

(e) For any fraction  $\frac{m}{n}$ , determine  $f\left(\frac{m}{n}\right)$ .

(f) For any rational number  $t$ , determine  $f(t)$ .

Note: If  $f$  is assumed to be “nice enough”, then for any real number  $t$ , the formula for  $f(t)$  is the same as in the case when  $t$  is any rational number.



## Exercise 5

The following data set is associated with a quantity that experiences exponential change. Express the associated model as a function of time.

$t$	$A(t)$
0	2
1	4
2	8
3	16



## Exercise 6

The following data set is associated with a quantity that experiences exponential change. Express the associated model as a function of time.

$t$	$A(t)$
3	2
6	4
9	8
12	16

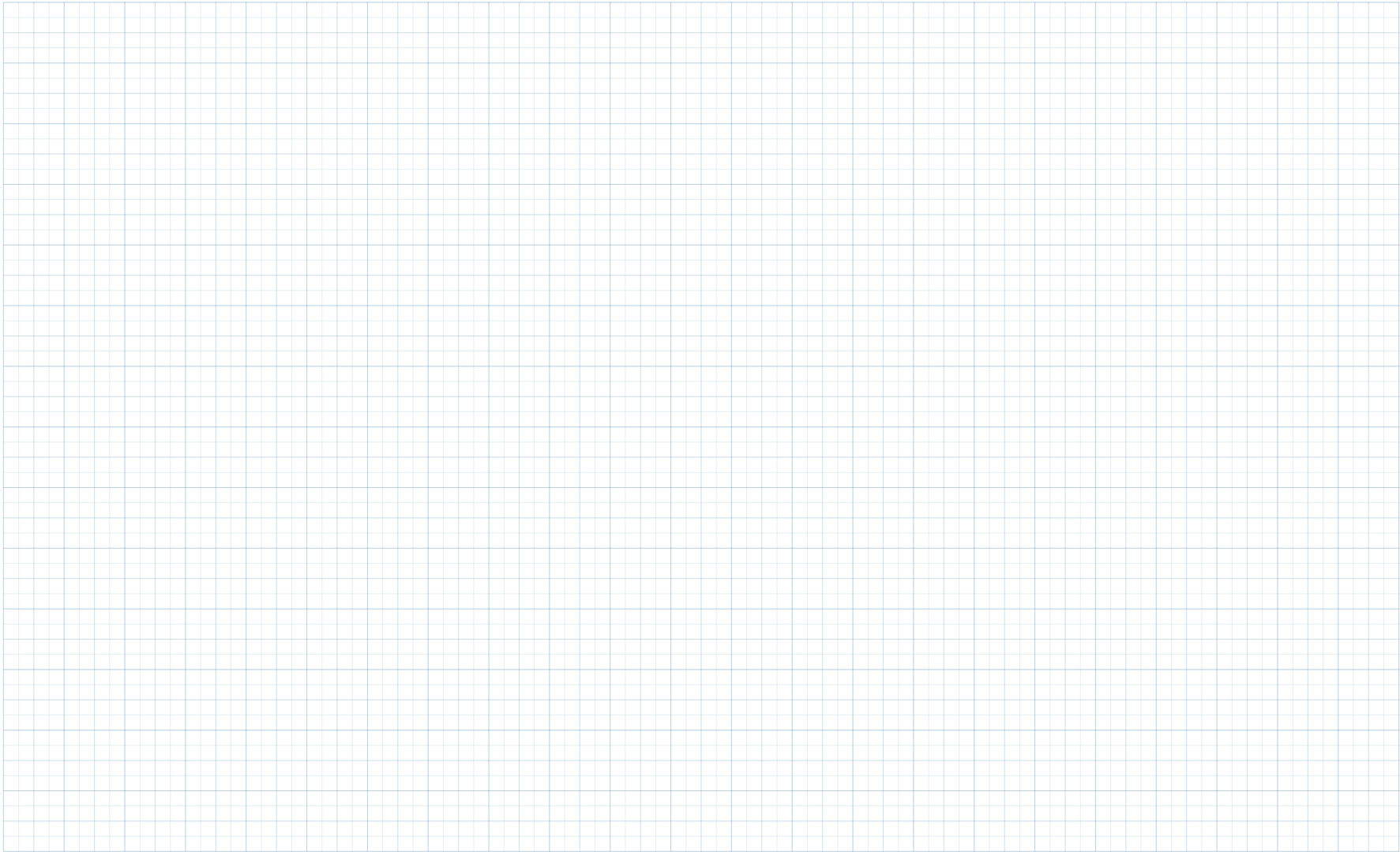
## Exercise 7

Determine all solutions to these equation:

(a)  $x - 5^{2x+1}x = 0$ ;

(b)  $\log_3(x) + \log_3(x - 2) = 1$ .

What errors in reasoning would you expect students to make that would introduce errors in their solutions?



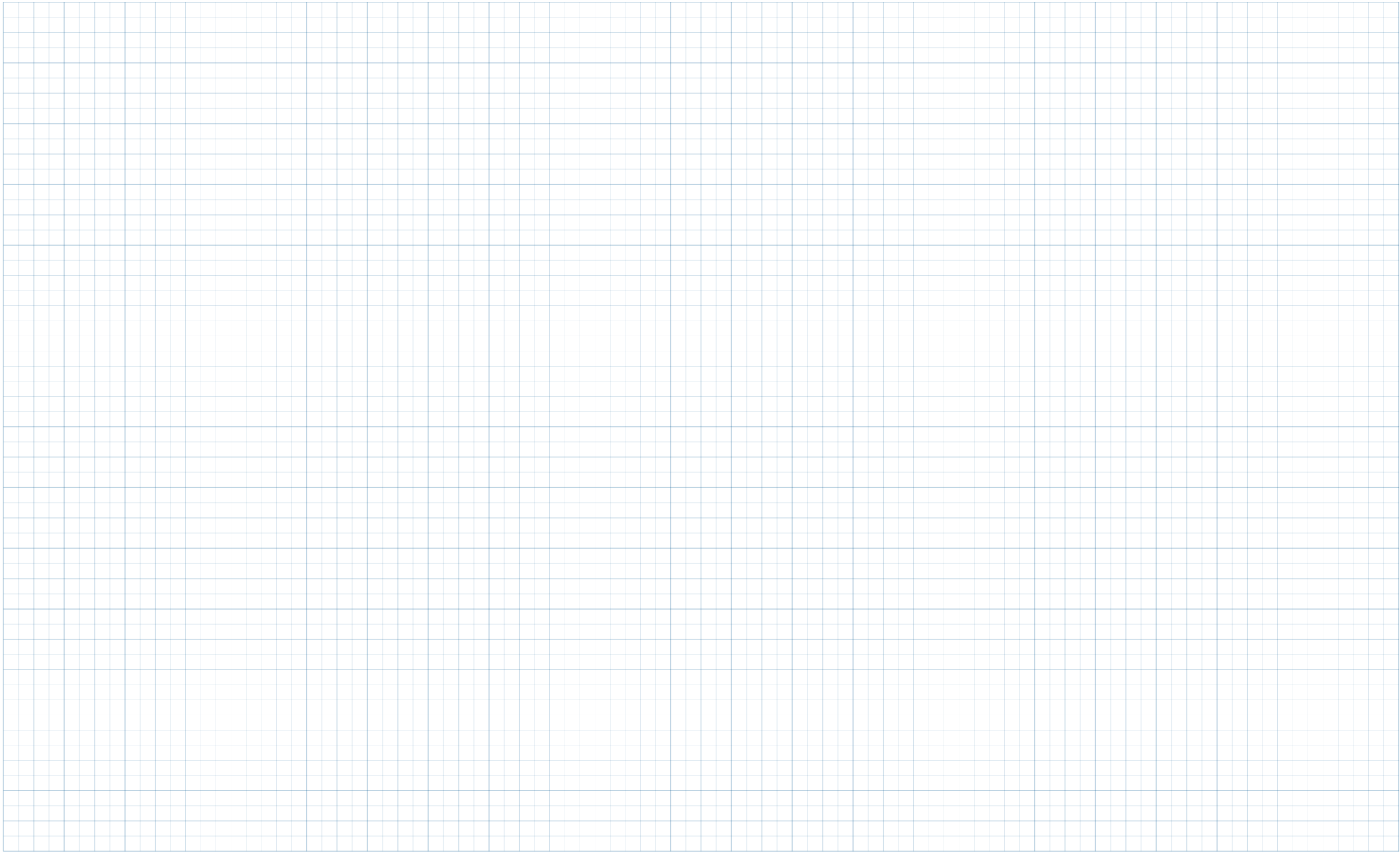
## Exercise 8

What is the decay rate (or growth rate) and the half-life (or doubling life) of a quantity that is described by an exponential growth model?

## Exercise 9

The mass of a body undergoes unrestricted growth and triples after 7 years. At time 4 years, the mass of the body is 10 grams.

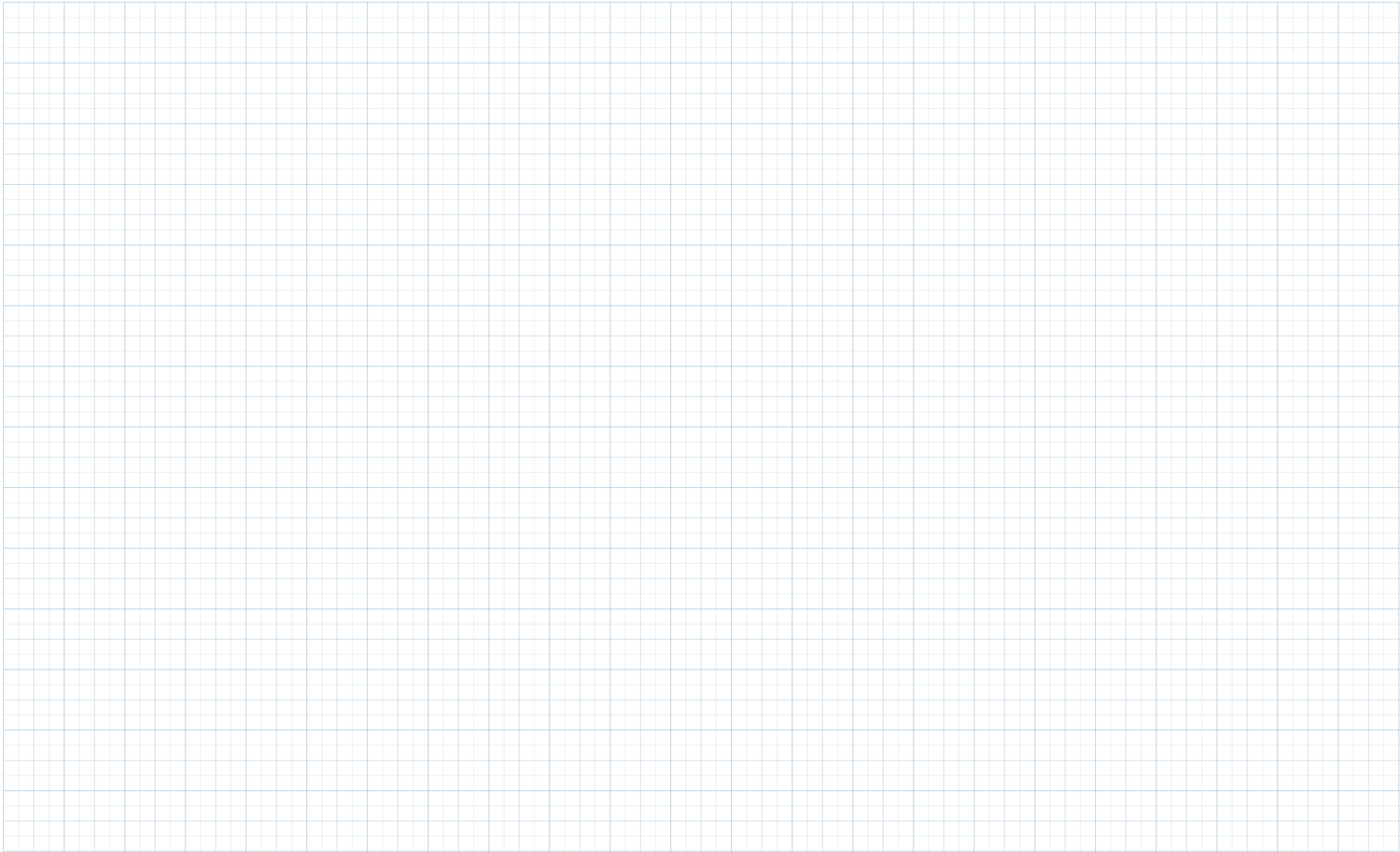
- (a) Identify a function that models the mass of the body, and determine the mass at time 0.
- (b) Identify the growth rate of the mass of the body.



## Exercise 10

A certain radioactive element has a half-life of 5 hours. You are given a sample of the material at time 0. At time 2 hours, 10 kilograms of the substance remains.

- (a) Identify a function that describes the amount of material that remains at time  $t$  hours after you were given the sample.
- (b) Determine the decay rate of the substance.





## Exercise 11

A certain radioactive substance has an unknown half-life. You are given a sample of 12 kilograms of the substance at time 0. After 2 days, 5 kilograms of the substance remain. Determine the half-life of the substance.

