Linguistic Mapping

The Principles of Calculus I

IV

Symmetry

IV.3

Symmetric Change

Classroom Exercises

Exercise 1

Two symmetries for the change that a quantity experiences are restrictive enough to uniquely determine the functions:

On any two intervals of the same length, the quantity changes by either

- the same **amount** (a linear model of change), or
- the same factor (an exponential model of change).

(a) Take f to be a function that changes linearly with respect to t, so that $f(t)$ is the numerical value of the quantity at t, M is the **amount** by which f changes between 0 and 1, and $f(0)$ is the numerical value of f at time 0. Identify a formula for $f(1)$.

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(b) Take a to be the amount by which f changes on $\left[0, \frac{1}{n}\right]$. Identify a formula for $f\left(\frac{1}{n}\right)$.

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(c) How does f change on each of the intervals $\left[0, \frac{1}{n}\right]$, $\left[\frac{1}{n}, \frac{2}{n}\right]$, $\left[\frac{2}{n}, \frac{3}{n}\right]$, and so on?

(d) Determine a in terms of M and n.

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(e) For any fraction $\frac{m}{n}$, determine $f\left(\frac{m}{n}\right)$.

(f) For any rational number t, determine $f(t)$.

Note: If f is assumed to be "nice enough", then for any real number t, the formula for $f(t)$ is the same as in the case when t is any rational number.

Exercise 2

The following data set is associated with a quantity that experiences linear change.

(a) Express the associated model as a function of time.

(b) Identify the minimal number of data points needed in this table to determine the model.

t	$A(t)$
0	4
1	8
2	12
3	16

Exercise 3

The following data set is associated with a quantity that experiences linear change. Express the associated model as a function of time.

t	$A(t)$
3	4
6	8
9	12
12	16

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(a) Take f to be a function that changes exponentially with respect to t, so that $f(t)$ is the
numerical value of the quantity at t, M is the **factor** by which f changes between 0 and 1, and $f(0)$ is the numerical value of f at time 0. Identify a formula for $f(1)$.

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Exercise 5

The following data set is associated with a quantity that experiences exponential change. Express the associated model as a function of time.

t	$A(t)$
0	2
1	4
2	8
3	16

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t	$A(t)$
3	2
6	4
9	8
12	16

Exercise 7

Determine all solutions to these equation:

(a) $x - 5^{2x+1}x = 0$;

(b) $\log_3(x) + \log_3(x-2) = 1$.

What errors in reasoning would you expect students to make that would introduce errors in their solutions?

Exercise 8

What is the decay rate (or growth rate) and the half-life (or doubling life) of a quantity that is described by an exponential growth model?

Exercise 9

The mass of a body undergoes unrestricted growth and triples after 7 years. At time 4 years, the mass of the body is 10 grams.

(a) Identify a function that models the mass of the body, and determine the mass at time 0.

(b) Identify the growth rate of the mass of the body.

Exercise 10

A certain radioactive element has a half-life of 5 hours. You are given a sample of the material at time 0. At time 2 hours, 10 kilograms of the substance remains.

(a) Identify a function that describes the amount of material that remains at time t hours after you were given the sample.

(b) Determine the decay rate of the substance.

Exercise 11

A certain radioactive substance has an unknown half-life. You are given a sample of 12 kilograms of the substance at time 0. After 2 days, 5 kilograms of the substance remain. Determine the half-life of the substance.