Linguistic Mapping

The Principles of Calculus I

IV

Symmetry

IV.2

Translational Symmetry

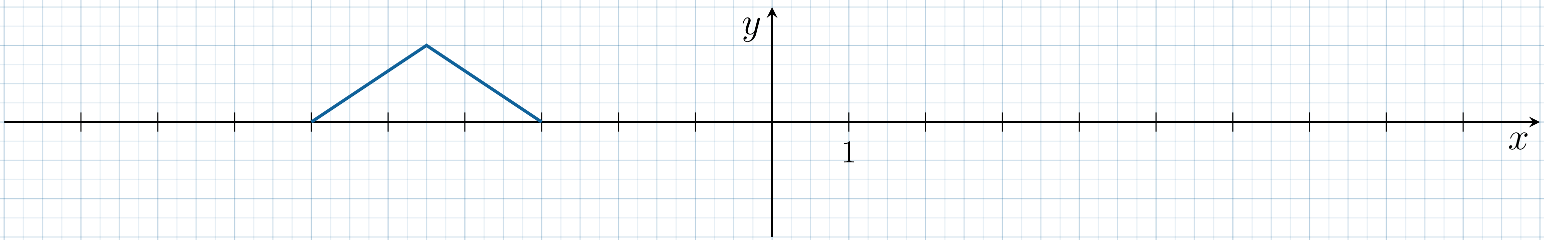
Classroom Exercises

Exercise 1

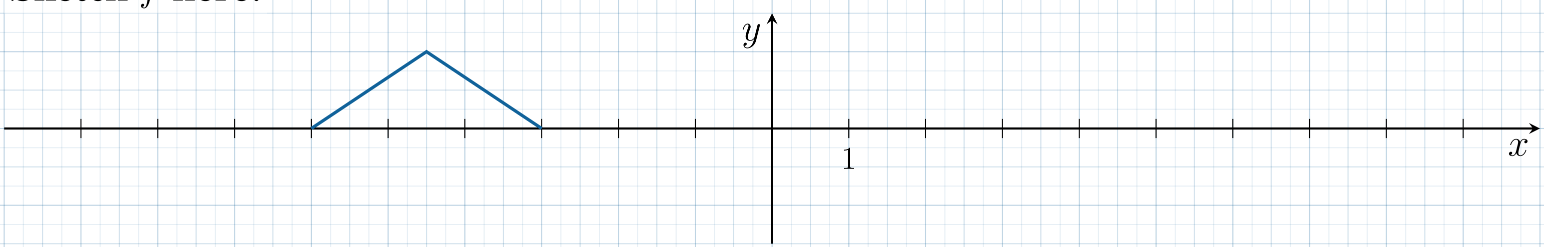
Take f to be a function for which translation by $\langle 6, 0 \rangle$ is a symmetry.

(a) Must f have a fundamental period?

(b) Given that f looks like this over $[-6, -3]$, and that f is an odd function, sketch f and determine whether f must have a fundamental period:



Sketch f here:



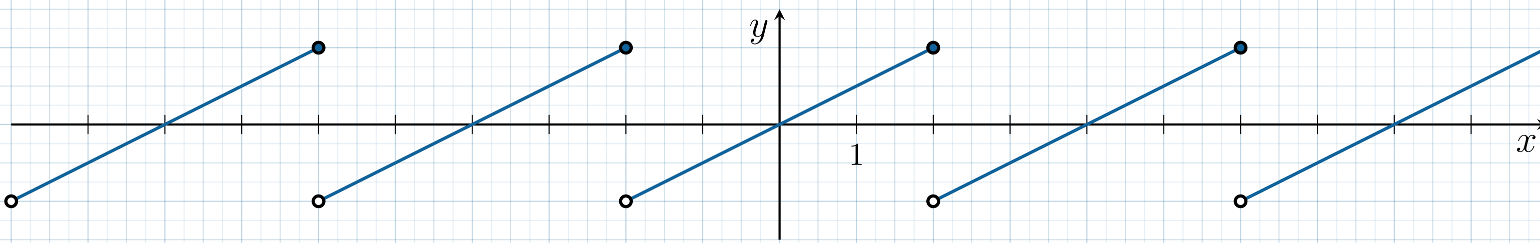
Exercise 2

Take f to be the periodic function whose sketch is given below and g to be given by

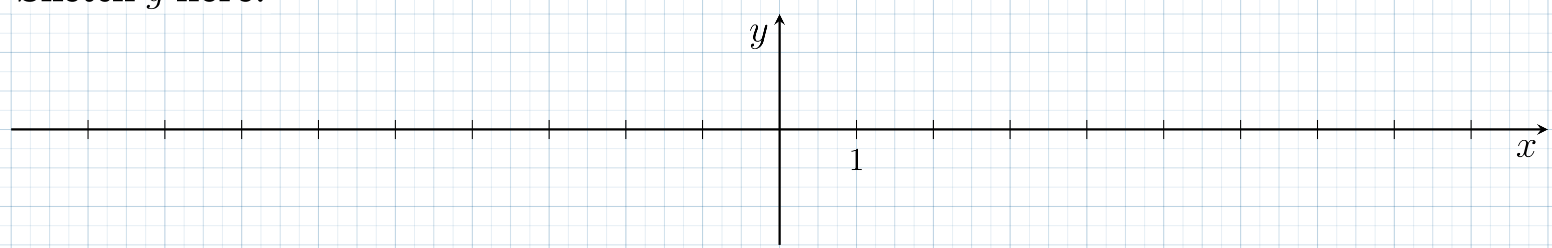
$$g(x) = (f \circ T_1)(x).$$

(a) Identify the fundamental period of f.

(b) Sketch g and determine the fundamental period of g.



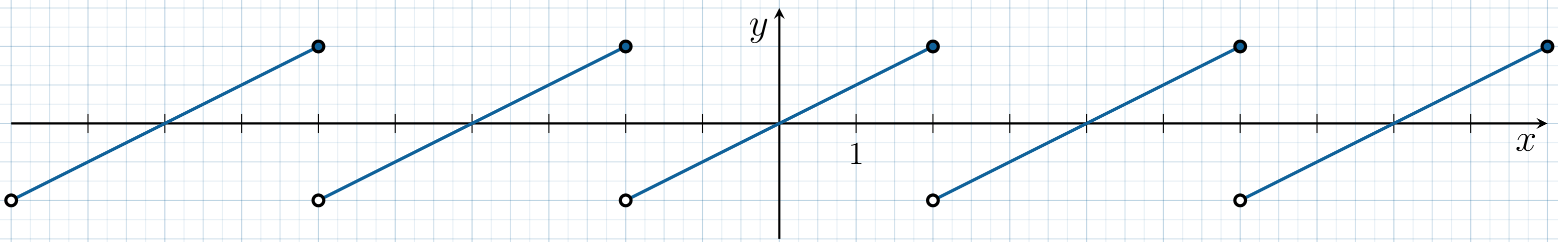
Sketch g here:



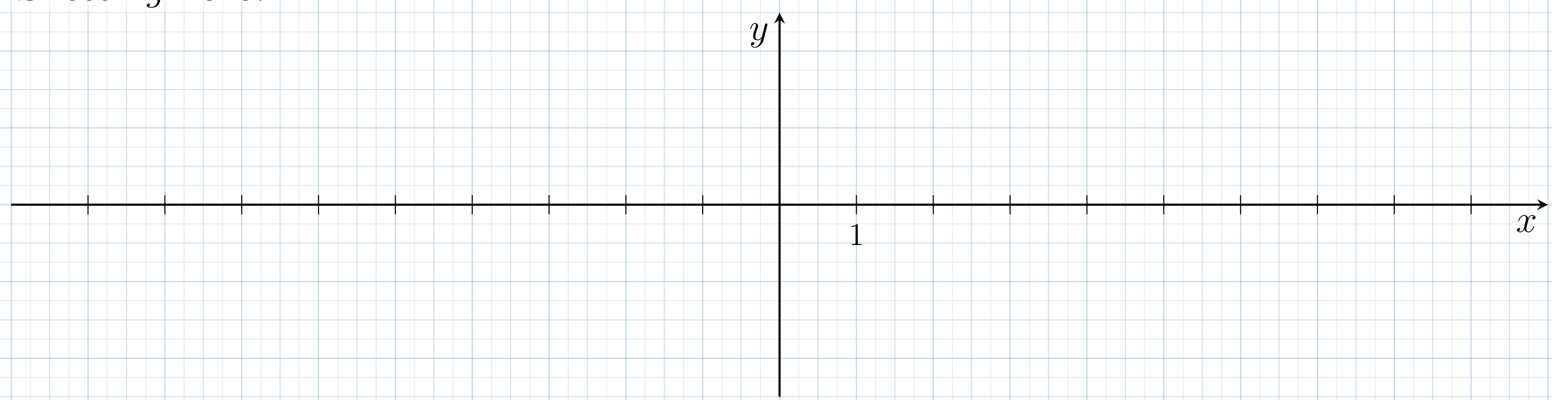
Take f to be the periodic function whose sketch is given below and g to be given by

$$g(x) = (S_2 \circ f)(x).$$

(c) Sketch g and determine the fundamental period of g.



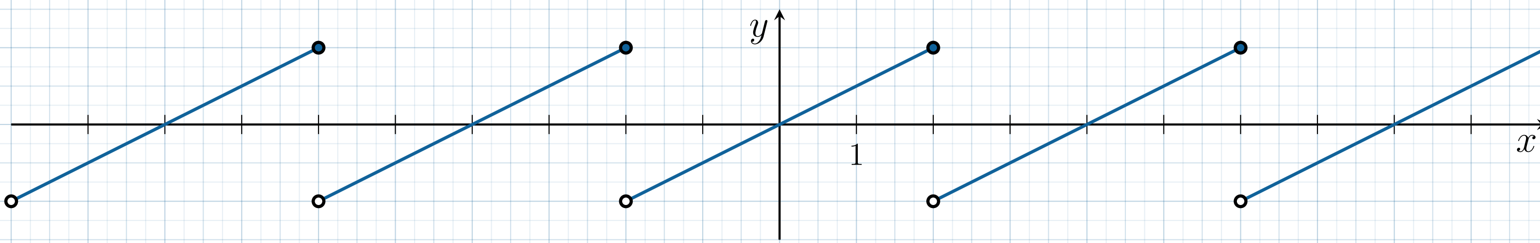
Sketch g here:



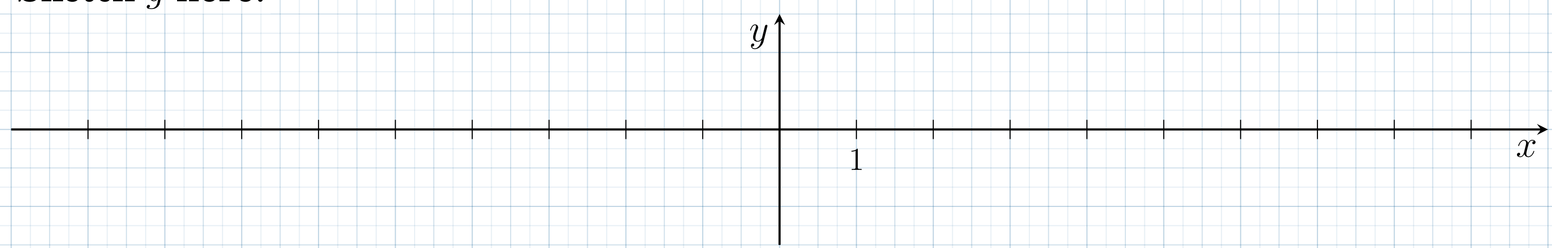
Take f to be the periodic function whose sketch is given below and g to be given by

$$g(x) = (f \circ S_2)(x).$$

(d) Sketch g and determine the fundamental period of g.



Sketch g here:

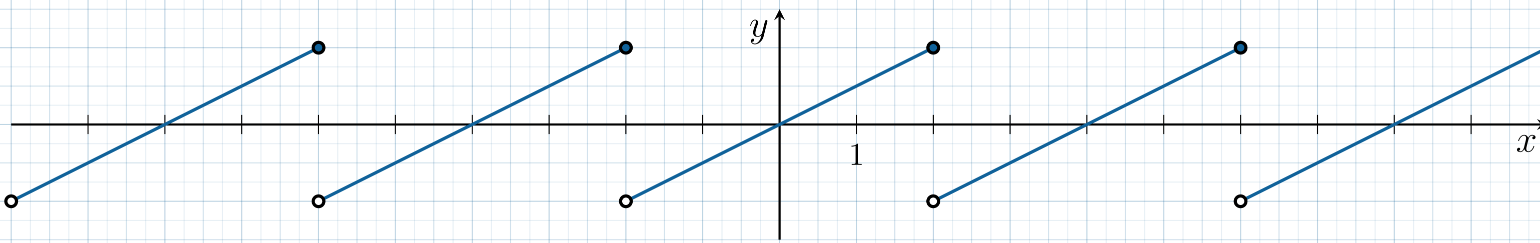


Take f to be the periodic function whose sketch is given below and g to be given by

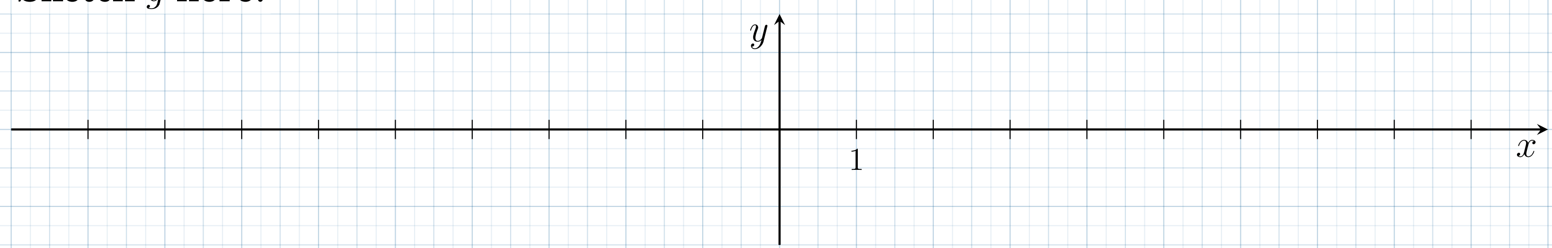
$$g(x) = f(2x + 1).$$

(e) Sketch g and determine the fundamental period of g.

(f) Explain the effects of translation and scaling on the fundamental period of f.



Sketch g here:



Exercise 3

Take f to be a function with fundamental period P. Take m to be a non-zero real number and b to be a real number. Take g to be the function given by

$$g(x) = mx + b.$$

(a) Explain why $f \circ g$ is a periodic function.

(b) Determine the fundamental period of $f \circ g$.

(c) Take f to be a function with fundamental period 3. Determine the fundamental period of the function g that is given by

$$g(x) = f(5x + 1).$$

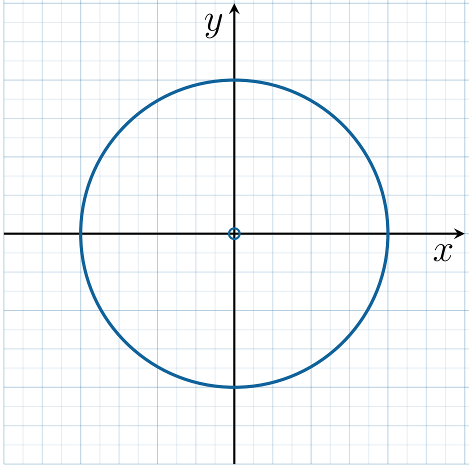
Exercise 4

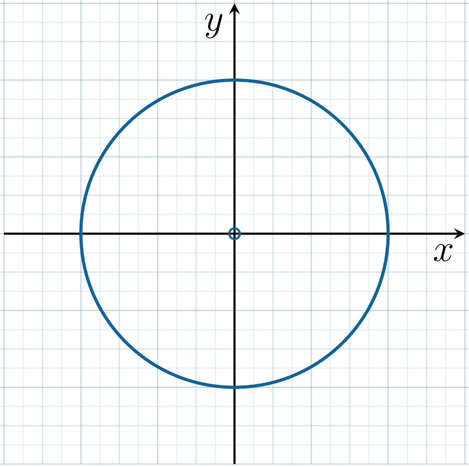
The trigonometric functions cos, sin, and tan are periodic functions.

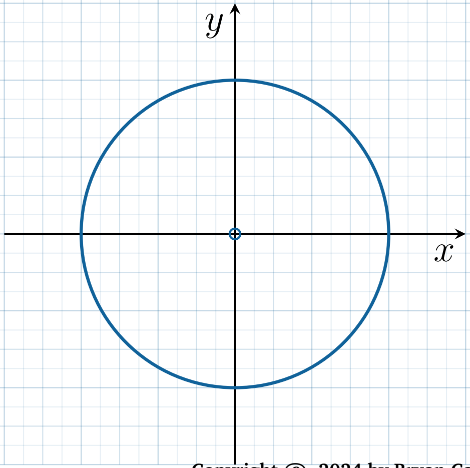
(a) Explain why cos and sin are periodic function with fundamental period 2π.

(b) Explain why tan is a periodic function with fundamental period π.

(c) Explain how symmetry can be used to sketch sine, cosine and tangent.







Exercise 5

No periodic function passes the horizontal line test.

(a) Explain in plain English the meaning of the horizontal line test.

(b) Explain why a periodic function cannot pass the horizontal line test.

(c) Explain why a periodic function is not invertible.

(d) Explain how to restrict a periodic function so that its restriction is invertible.

Exercise 6

Take S to be the square with vertices $(1,-1)$, $(1,1)$, $(-1,1)$, and $(-1,-1)$. Take f to be the function that for each (x,y) in $\mathbb{R}^2$ is given by

$$f(x,y) = x.$$

For each t in $\mathbb{R}$, there is a unique point $(x(t),y(t))$ in an edge of S that is the position of a particle that moves counterclockwise along the edges of S a distance of t from $(1,0)$ if t is non-negative, and that moves clockwise along the edges of S a distance of $|t|$ from $(1,0)$ if t is negative.

(a) Take g to be the function that is given by

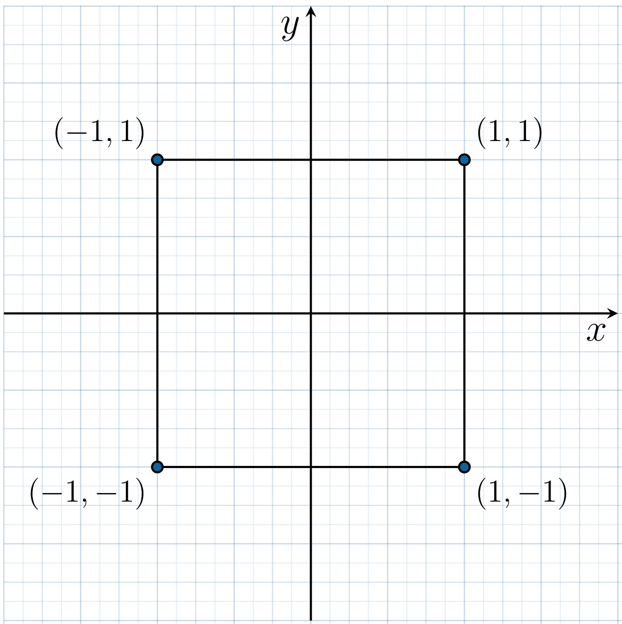
$$g(t) = f((x(t),y(t))).$$

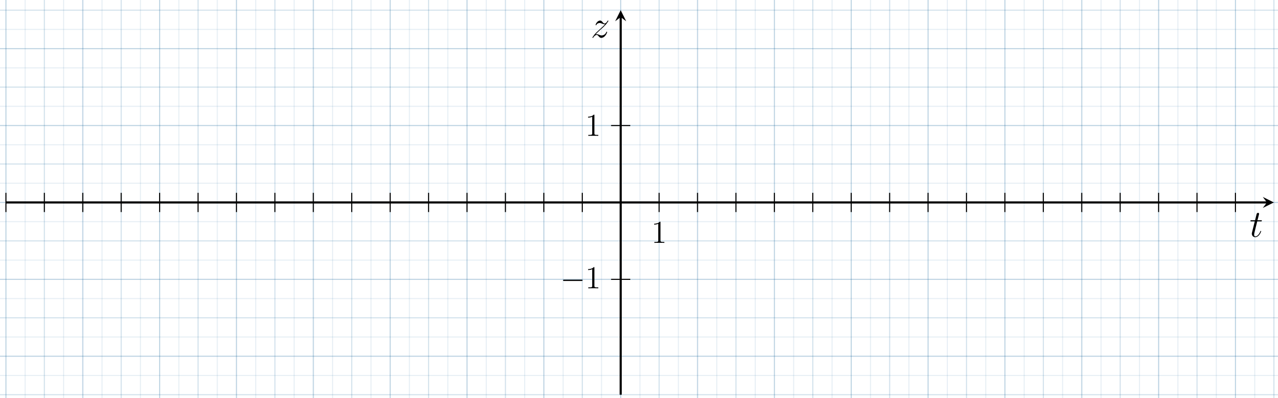
Sketch g, identify all periods of g, and identify the fundamental period of g.

(b) Identify all intersections of g with the horizontal line that intersects $(0,\frac{2}{3})$.

(c) Identify all solutions to the equation

$$g(x) = \frac{2}{3}.$$


y
x
(−1, 1)
(1, 1)
(−1, −1)
(1, −1)


z
t
1
1
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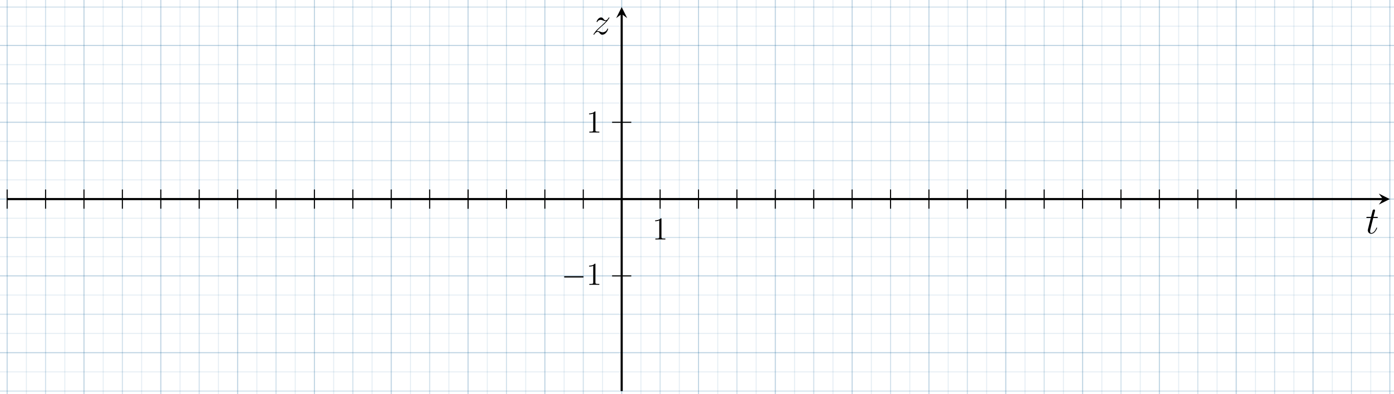
z
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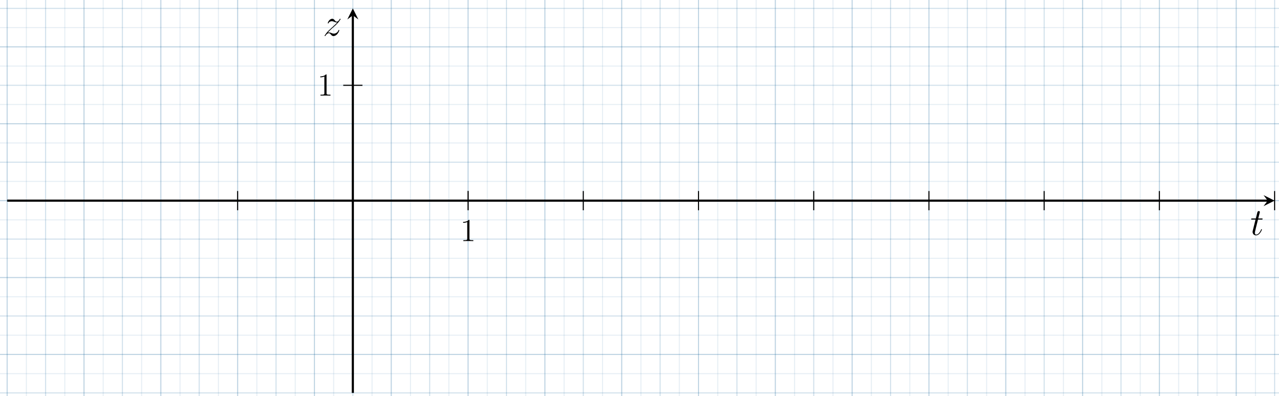

z
1
1
t

Exercise 7

Take Cos, Sin, **and** Tan **to be the** ***principle*** **cosine, sine, and tangent functions, respectively, so that**

$$\text{Cos} = \cos\Big|_{[0,\pi]}, \quad \text{Sin} = \sin\Big|_{[-\frac{\pi}{2},\frac{\pi}{2}]}, \quad \textbf{and} \quad \text{Tan} = \tan\Big|_{(-\frac{\pi}{2},\frac{\pi}{2})},$$

and denote by $\arccos$, $\arcsin$, **and** $\arctan$ **the functions**

$$\arccos = \text{Cos}^{-1}, \quad \arcsin = \text{Sin}^{-1}, \quad \textbf{and} \quad \arctan = \text{Tan}^{-1}.$$

(a) Determine the domain of $\arccos$, $\arcsin$, **and** $\arctan$.

(b) Determine the range of $\arccos$, $\arcsin$, **and** $\arctan$.

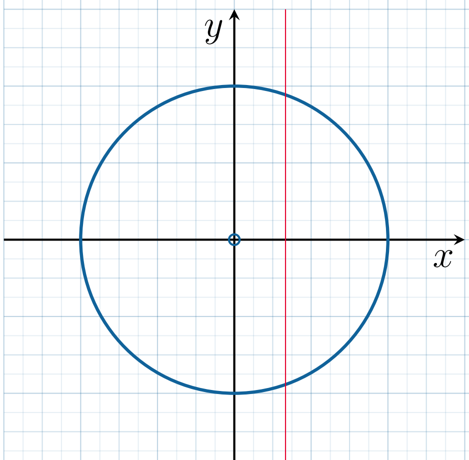
(c) Determine $\cos^{-1}(\frac{1}{2})$ **and** $\arccos(\frac{1}{2})$.

Exercise 8

Sketch all points on the unit circle whose x-coordinate is $\frac{1}{3}$ and associate these points to angle measures.

(a) Identify all solutions that lie in $[0, 2\pi)$ to the equation

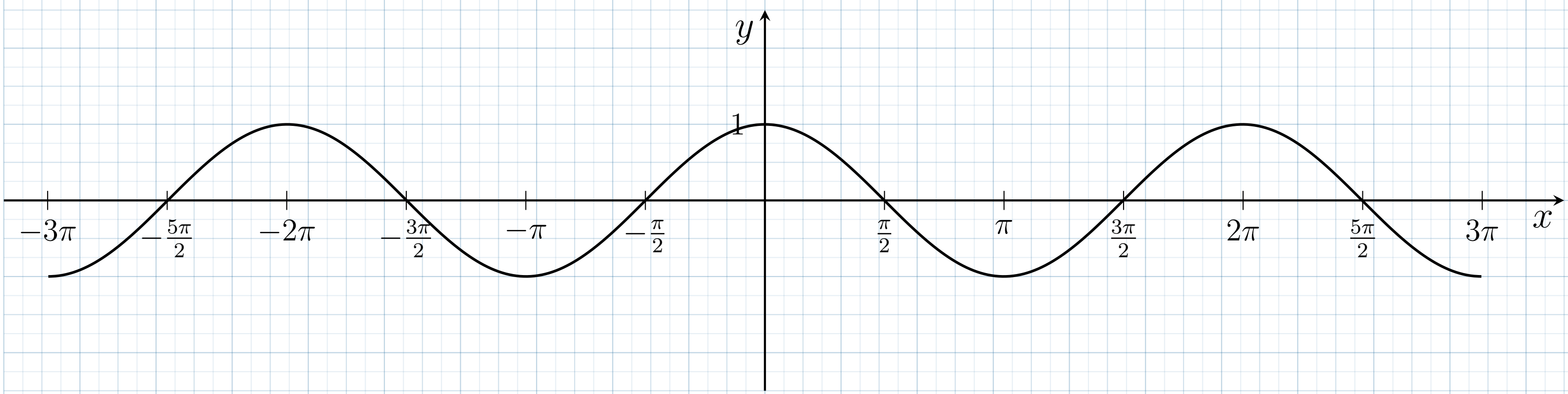
$$\cos(x) = \frac{1}{3}.$$



(b) Identify all solutions to the equation

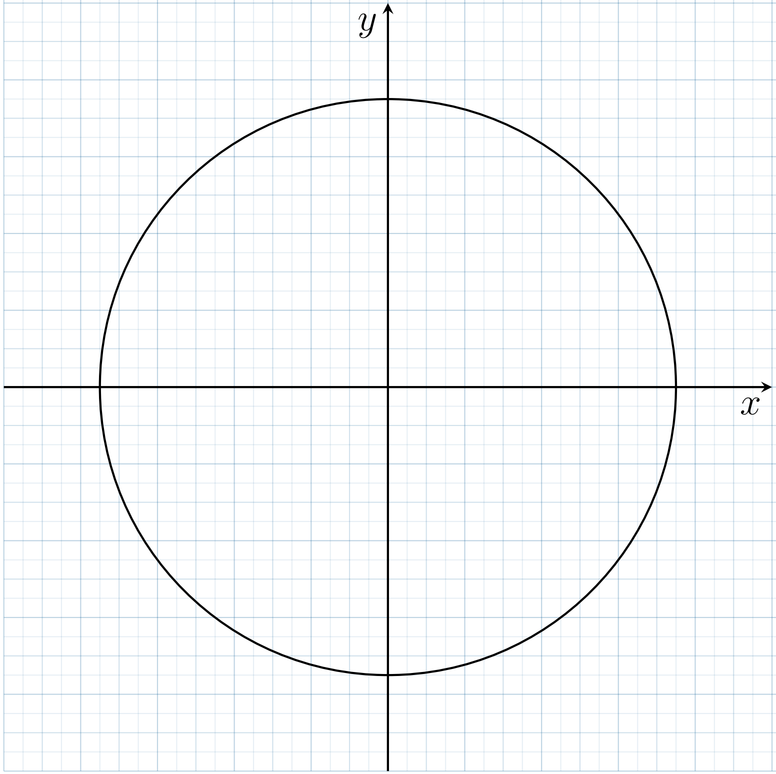
$$\cos(x) = \frac{1}{3},$$

and sketch these solutions on the sketch of cos given below.



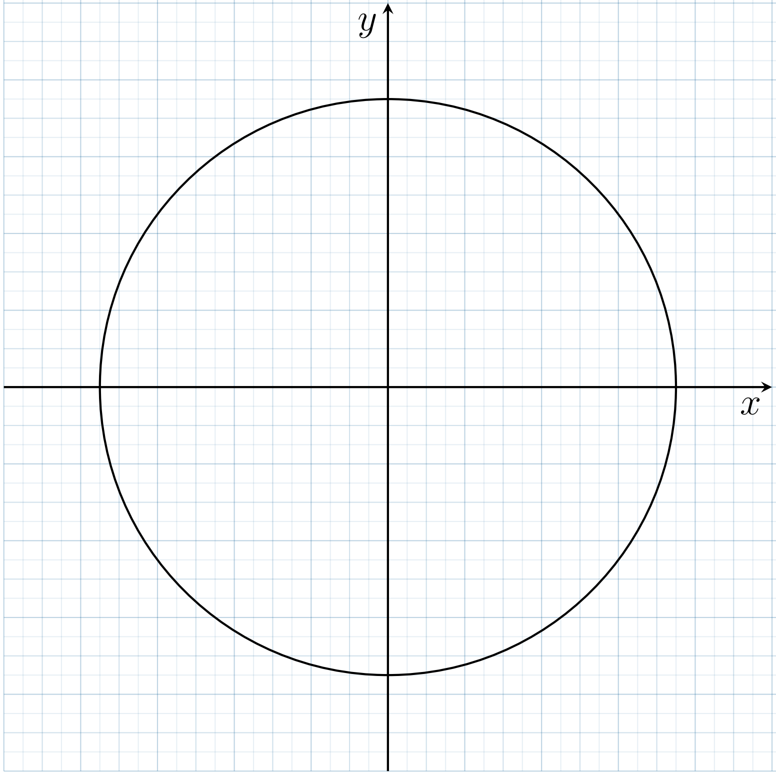
Exercise 10

Evaluate the quantity $\arcsin\left(\sin\left(\frac{\pi}{9}\right)\right)$.



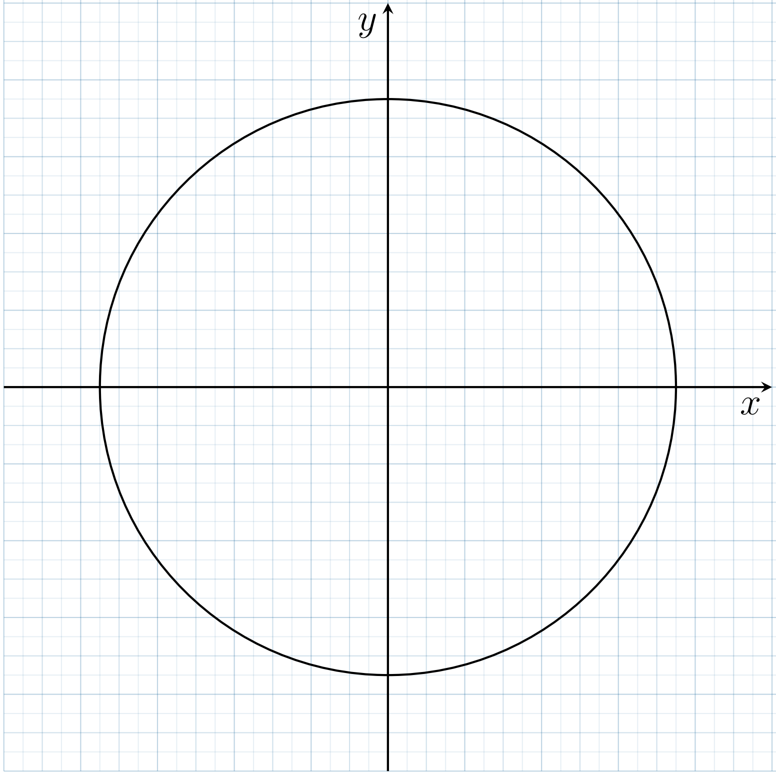
Exercise 11

Evaluate the quantity $\arctan\left(\tan\left(-\frac{\pi}{9}\right)\right)$.



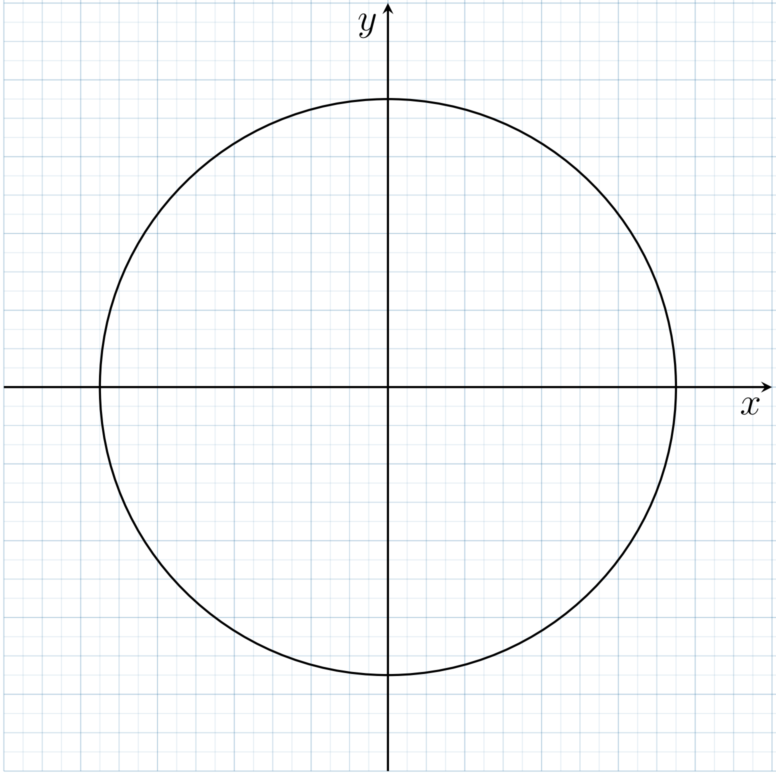
Exercise 12

Evaluate the quantity $\arccos\left(\cos\left(\frac{7\pi}{9}\right)\right)$.



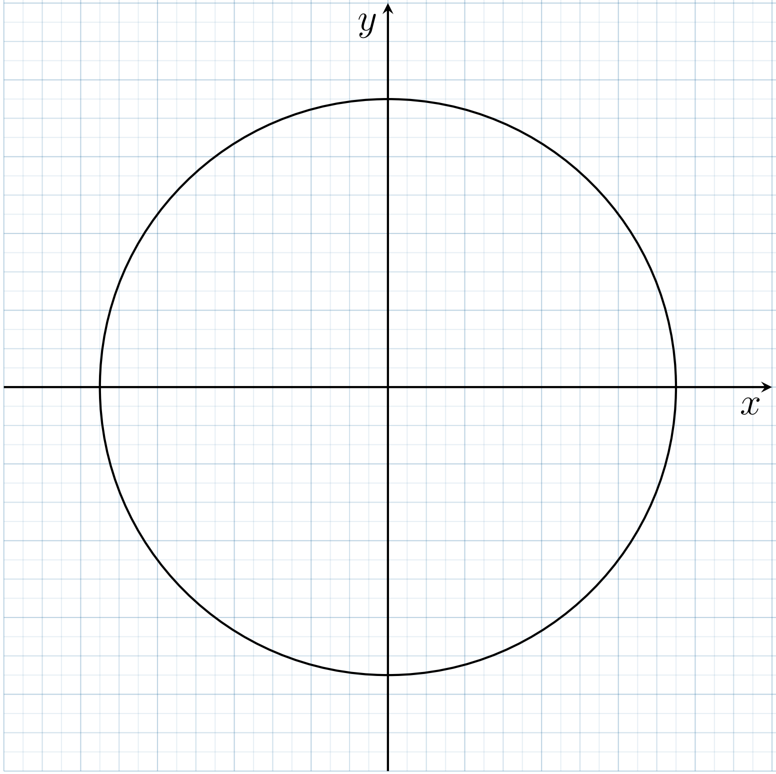
Exercise 13

Evaluate the quantity $\arcsin\left(\sin\left(\frac{7\pi}{5}\right)\right)$.



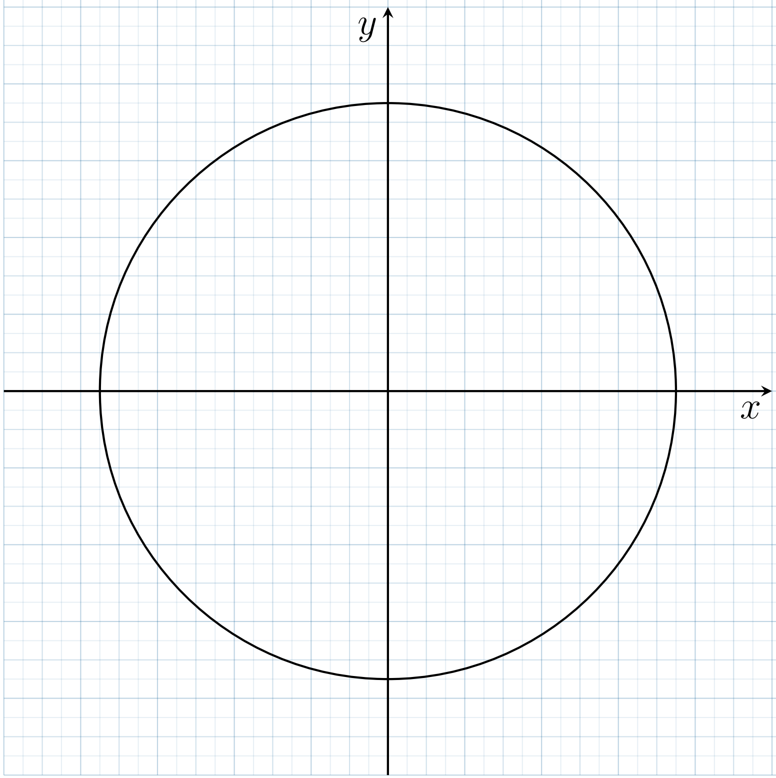
Exercise 14

Evaluate the quantity $\arctan\left(\tan\left(\frac{11\pi}{8}\right)\right)$.



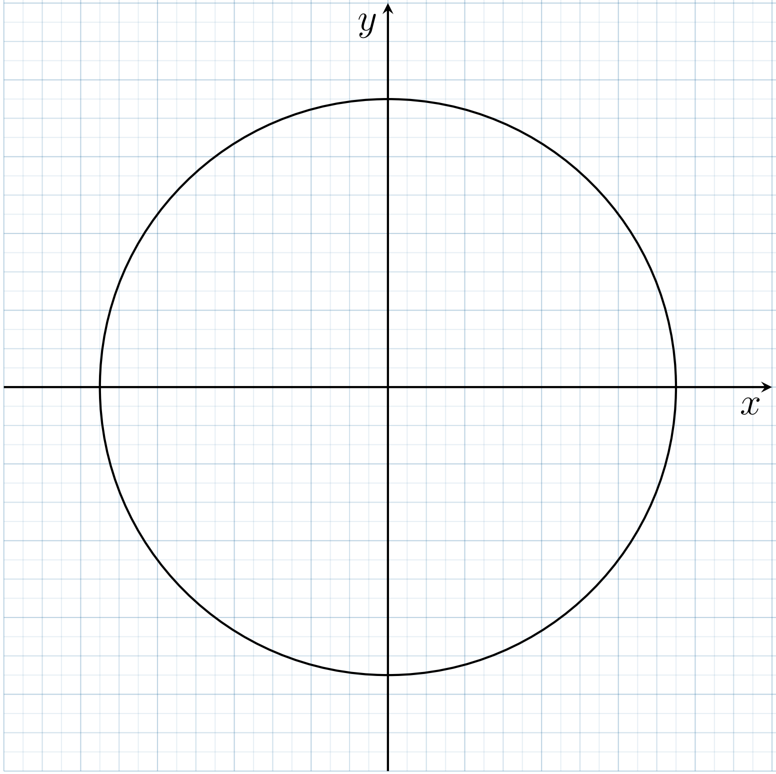
Exercise 15

Evaluate the quantity $\arccos\left(\cos\left(-\frac{\pi}{9}\right)\right)$.



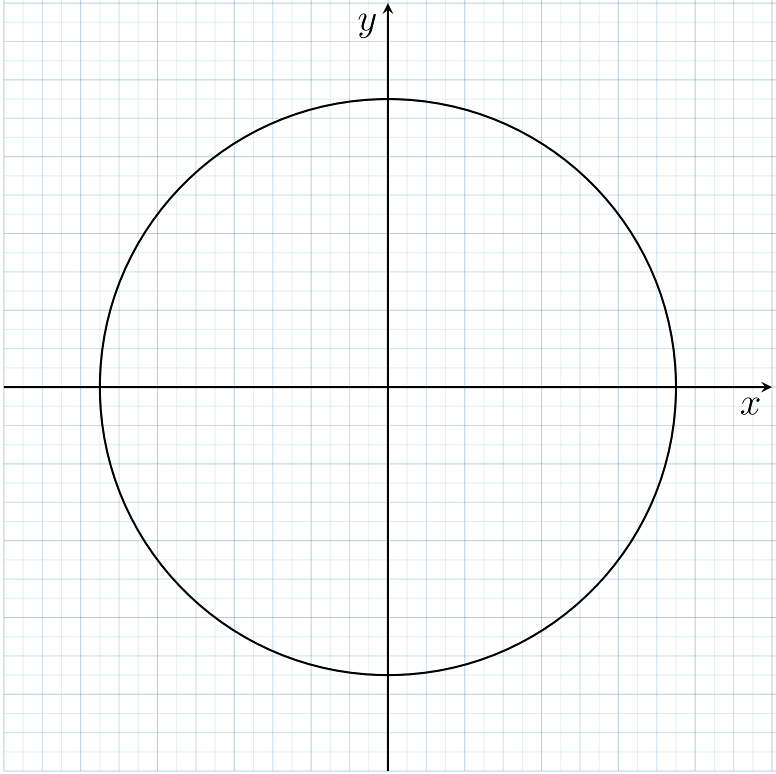
Exercise 16

Evaluate the quantity $\tan\left(\arcsin\left(\frac{1}{5}\right)\right)$.



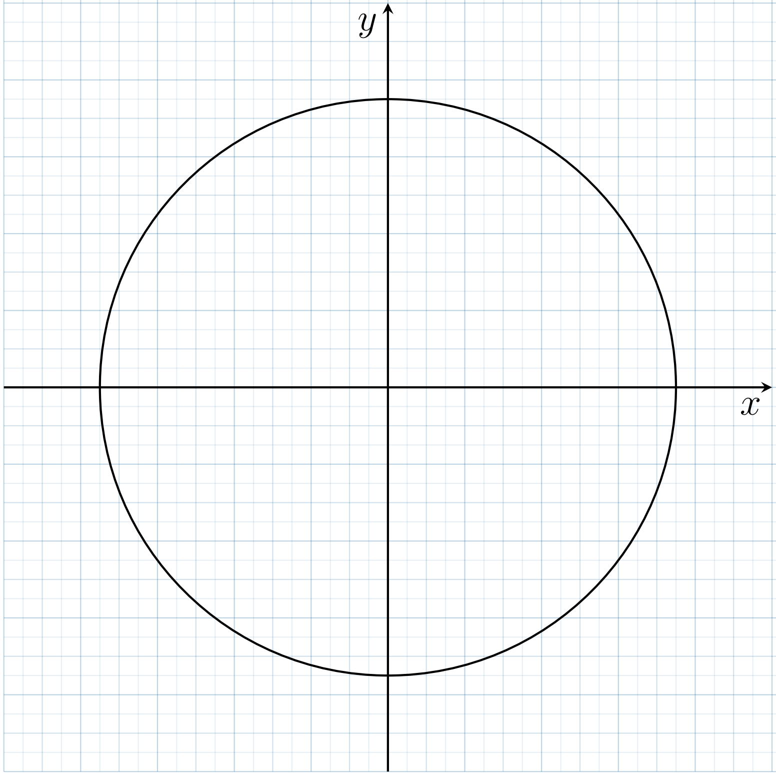
Exercise 17

Evaluate the quantity $\arcsin\left(\cos\left(\frac{8\pi}{7}\right)\right)$.



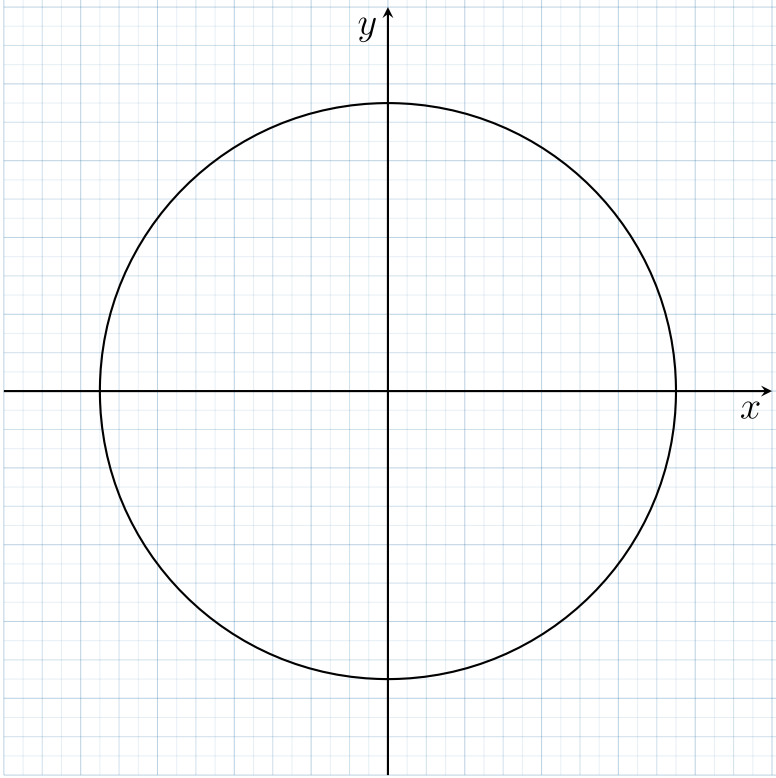
Exercise 18

Evaluate the quantity $\cos(\arcsin(5a-1))$. For what values of a will this make sense?



Exercise 19

Evaluate the quantity $\sin\left(\arctan\left(a^2+9\right)\right)$. For what values of a will this make sense?



Exercise 20

Some trigonometric equations can be solved exactly. Symmetry is useful in writing out the solution sets.

(a) Determine the solution set to this equality for θ in $[0, 2\pi]$:

$$\cot^2(\theta) = 3.$$

(b) Use symmetry to determine the solution set to this equality for any θ:

$$\cot^2(\theta) = 3.$$

Using transformation, rigidity and symmetry together is useful for solving more complicated problems. Consider the following equality:

$$5\sin^2(3\theta - 1) + 29\sin(3\theta - 1) - 6 = 0.$$

(a) Denote by y the the quantity $\sin(3\theta - 1)$. Rewrite the above equality so it is a polynomial inequality in variable y.

(b) Solve the polynomial equation in variable y. Call the solutions y_1 and y_2.

Using transformation, rigidity and symmetry together is useful for solving more complicated problems. Consider the following equality:

$$5\sin^2(3\theta-1)+29\sin(3\theta-1)-6=0.$$

(c) Replace y with $\sin(3\theta-1)$ and $(3\theta-1)$ with ϕ. If possible solve the equations

$$\sin(\phi)=y_1 \quad \textbf{or} \quad \sin(\phi)=y_2,$$

where ϕ is in $[0,2\pi]$.

Using transformation, rigidity and symmetry together is useful for solving more complicated problems. Consider the following equality:

$$5\sin^2(3\theta - 1) + 29\sin(3\theta - 1) - 6 = 0.$$

(d) Scale the solution sets in (c) appropriately to get the solution set for the original equality.

(e) Explain clearly how you used the principles of decomposition, transformation, rigidity, and symmetry to determine the solution set.