Linguistic Mapping

The Principles of Calculus I

IV

Symmetry

IV.1

Introduction to Symmetry

Classroom Exercises

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Exercise 1

Take $(\mathscr{C}, \star)$ to be the unit circle paired with the binary operation given by

$$(a, b) \star (c, d) = (ac - bd, ad + bc).$$

(a) Explain in plain English what “binary operation” means and show that $\star$ is a binary operation on $\mathscr{C}$.

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Take $(\mathscr{C}, \star)$ to be the unit circle paired with the binary operation given by

$$(a, b) \star (c, d) = (ac - bd, ad + bc).$$

(b) Explain in plain English what it means for a binary operation to be associative and show that $\star$ is an associative binary operation on $\mathscr{C}$.

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Take $(\mathscr{C}, \star)$ to be the unit circle paired with the binary operation given by

$$(a, b) \star (c, d) = (ac - bd, ad + bc).$$

(c) Explain in plain English what an identity element is and identify the identity element in $(\mathscr{C}, \star)$.

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Take $(\mathscr{C}, \star)$ to be the unit circle paired with the binary operation given by

$$(a, b) \star (c, d) = (ac - bd, ad + bc).$$

(d) Explain in plain English what it means for an element to be the inverse of another element.

(e) Given an element p in $(\mathscr{C}, \star)$, identify its inverse p^{-1}.

(f) Explain why $(\mathscr{C}, \star)$ is a group.

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Exercise 2

Identifying the symmetry group of a set with a certain structure can help us to better understand and determine the properties of the set and the structure. For example, the group $(H, \bullet)$ formed by the translations of the plane together with the non-zero scalings of the y-axis is a group that preserves the set of quadratic polynomials.

(a) Explain what this statement means.

(b) For any two elements g_1 and g_2 of H, define the element $g_1 \bullet g_2$ by the equality

$$(g_1 \bullet g_2)f = g_1(g_2 f)$$

for any quadratic polynomials f. Explicitly write $Y_{c_2} \bullet \langle a_2, b_2 \rangle$ in terms of $\langle a_1, b_1 \rangle \bullet Y_{c_1}$.

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For example, the group $(H, \bullet)$ formed by the translations of the plane together with the non-zero scalings of the y-axis is a group that preserves the set of quadratic polynomials.

(c) Can any element of H be written as $\langle a, b\rangle \bullet Y_c$ for suitable choices of a, b, and c?

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For example, the group H formed by the translations of the plane together with the non-zero scalings of the y-axis is a group that preserves the set of quadratic polynomials. For any real number a and any nonzero real number b, denote

$$T_a(x) = x + a \quad \textbf{and} \quad S_b(x) = bx.$$

(d) Show that for any element h in H there are real numbers A, B, and C so that for any quadratic polynomial f,

$$hf = T_A \circ S_B \circ f \circ T_C.$$

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Exercise 3

Take $(H, \bullet)$ to be the group formed by the translations of the plane together with the non-zero scalings of the y-axis. Take f and g to be the quadratic polynomials given by

$$f(x) = 4x^2 + 5x + 1 \quad \text{and} \quad g(x) = 5x^2 + 3x + 2.$$

(a) Identify the vertex of f, the vertex of $Y_{\frac{5}{4}}f$, and the vertex of g.

(b) Determine an element of H that takes f to g.

(c) Use the rigidity of quadratic polynomials to determine the symmetry that takes f to g, and carefully explain how use of this rigidity simplified the problem.

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Exercise 4

Take S to be the group that consists of the identity, and is generated by reflection across the y-axis, and rotation by half a circle.

(a) Explain why S is a group and identify all elements of S.

(b) Describe in plain English the symmetries of an even function and the symmetries of an odd function.

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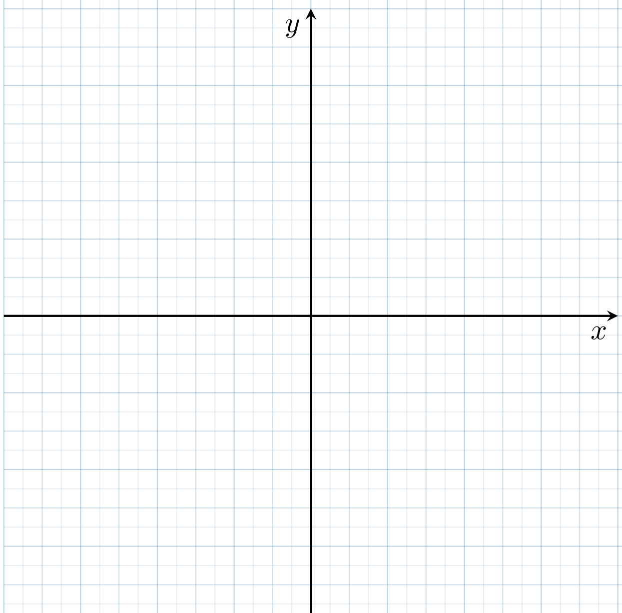
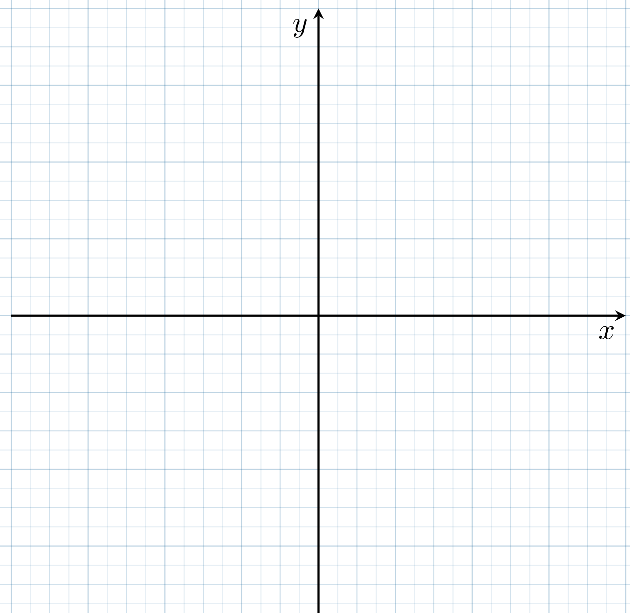
Take S to be the group that consists of the identity, and is generated by reflection across the y-axis, and rotation by half a circle.

(c) Sketch an example of an even function and an odd function.





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Exercise 6

Explain how to determine if a function is even or odd.

(a) Determine whether the function f is an even or odd function, where

$$f(x) = x^2 + x^4.$$

(b) Determine whether the function g is an even or odd function, where

$$g(x) = |x|x^3.$$

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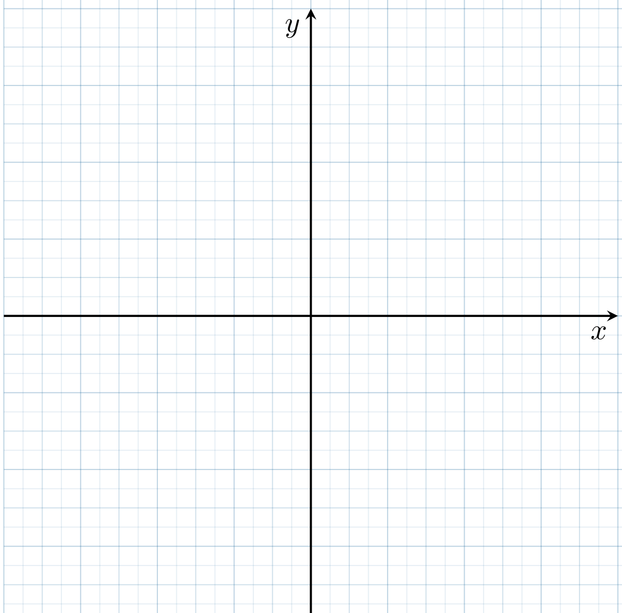
Exercise 7

Explain how to determine if a function is neither even nor odd.

(a) Sketch an example of a function that is neither even nor odd.

(b) Explain why the following function is neither even nor odd:

$$h(x) = x^3|x| + x^2.$$



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Exercise 8

For each choice of function f, explain why it is even, odd or neither:

(a) $f(x) = \frac{1}{x} + x$;

(b) $f(x) = \begin{cases} -x^3 & \text{if } x \leq 0 \\ x^3 & \text{if } x > 0; \end{cases}$

(c) $f(x) = -\frac{1}{x^2+1}$;

(d) $f(x) = \begin{cases} -\frac{1}{x^2+1} & \text{if } x \leq 0 \\ \frac{1}{x^2+1} & \text{if } x > 0. \end{cases}$

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Exercise 9

For each choice of function f, explain why it is even, odd or neither:

(a) $f(x) = \sin(x)$;

(b) $f(x) = \cos(x)$;

(c) $f(x) = \tan(x)$.

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Exercise 10

Functions with other symmetries are also possible. For example, a function may be symmetric under reflection across a vertical line that passes through $(a, 0)$ or under rotation by half a circle around $(a, 0)$. For each choice of function f, identify the symmetries of f:

(a) $f(x) = (x + 2)^4 + |x + 2|$;

(b) $f(x) = (x + 2)^4 + \frac{1}{x^2+4x+4}$;

(c) $f(x) = (x - 1)^3 - x + 3$.

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