Linguistic Mapping

# The Principles of Calculus I

## III

## Rigidity

## III.1

## Polynomial Functions

Classroom Exercises

# Exercise 1

The collection of quadratic polynomial functions is a rigid collection because any quadratic polynomial function is equal to $\mathrm{pow}_2$ up to a $y$-axis scaling and a translation of the plane.

What does this statement precisely mean?

For any real numbers $A$, $h$, and $k$ so that $A$ is nonzero, identify a formula for the quadratic polynomial $g$ that is given by

$$g = \langle h, k \rangle + Y_A \operatorname{pow}_2 .$$

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$$g = \langle h, k \rangle + Y_A \, \mathrm{pow}_2 \, .$$

(a) Identify the vertex of $g$.

(b) What relationship does the vertex of $g$ have with the extremal values of $g$?

For any real numbers $A$, $h$, and $k$ so that $A$ is nonzero, identify a formula for the quadratic polynomial $g$ that is given by

$$g = \langle h, k \rangle + Y_A \, \mathrm{pow}_2 \, .$$

(c) When will $g$ have zeros and how does this relate to the discriminant of $g$?

For any real numbers $A$, $h$, and $k$ so that $A$ is nonzero, identify a formula for the quadratic polynomial $g$ that is given by

$$g = \langle h, k \rangle + Y_A \, \mathrm{pow}_2 \, .$$

(d) Given that $a$ is nonzero, identify values for $A$, $h$, and $k$ so that

$$g(x) = ax^2 + bx + c.$$

# Exercise 2

Take $f$ and $g$ to be the polynomials that are given by

$$f(x) = 3x^2 - 6x + 15 \quad \textbf{and} \quad g(x) = -2x^2 - 12x + 14.$$

(a) Write $f$ and $g$ in a way that makes it clear how they arise from $\text{pow}_2$.

Take $f$ and $g$ to be the polynomials that are given by

$$f(x) = 3x^2 - 6x + 15 \quad \textbf{and} \quad g(x) = -2x^2 - 12x + 14.$$

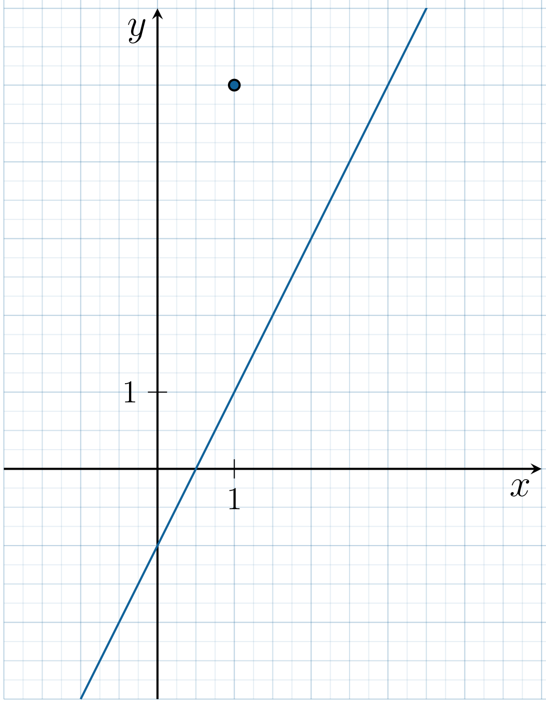
(b) Use the expressions for $f$ and $g$ in part (a) to determine the zeros of $f$ and $g$, the vertex of both $f$ and $g$.

(c) Identify and classify the extremal values for $f$ and $g$.

Take $L$ to be the line that is given by the equation

$$y = 2x - 1.$$

**(a)** For any $(x, y)$ in $L$, identify the vector $V(x)$ that moves the point $(1, 5)$ to the point $(x, y)$. Draw a picture to represent $V(x)$.



Take $L$ to be the line that is given by the equation

$$y = 2x - 1.$$

(b) For any $(x, y)$ in $L$, take $d(x)$ to be the distance from $(x, y)$ to $(1, 5)$. Identify a formula for $d(x)$ and $d(x)^2$.

Take $L$ to be the line that is given by the equation

$$y = 2x - 1.$$

**(c)** Do $d(x)$ and $d(x)^2$ take on their minimal values at the same points in their domain?

**(d)** Without appealing to geometry, determine the point on $L$ that is closest to $(1, 5)$.

## Exercise 4

Boat A and Boat B both move with constant velocities in the plane, Boat A with velocity $\langle 2, 4\rangle$ and Boat B with velocity $\langle -1, 2\rangle$. At time $0$, Boat A is at $(-4, -2)$ and Boat B is at $(6, 1)$. Take $A(t)$ and $B(t)$ to be the position for Boat A and Boat B, respectively, at time $t$.

(a) Determine formulas for $A(t)$ and $B(t)$.

(b) For each time $t$, identify the vector $V(t)$ that points from $B(t)$ to $A(t)$. What is the length of this vector?

Boat A and Boat B both move with constant velocities in the plane, Boat A with velocity $\langle 2, 4\rangle$ and Boat B with velocity $\langle -1, 2\rangle$. At time $0$, Boat A is at $(-4, -2)$ and Boat B is at $(6, 1)$. Take $A(t)$ and $B(t)$ to be the position for Boat A and Boat B, respectively, at time $t$.

(c) Determine the distance $d(t)$ between the two boats at time $t$.

(d) Determine the time at which the boats are closest to each other as well as their minimal distance.

# Exercise 5

To quickly determine the product

$$P(x) = (2x^3 + 3x^2 - x + 4)(5x^4 - x^3 + 3x^2 - 2x + 1),$$

answer the following questions:

(a) What is the degree of the product?

(b) $P$ is a sum of monomials. What are the possible degrees of these monomials?

To quickly determine the product

$$P(x) = (2x^3 + 3x^2 - x + 4)(5x^4 - x^3 + 3x^2 - 2x + 1),$$

answer the following questions:

(c) Write $P$ as a sum of monomials, but where the coefficients are left like this:

$$\Big( \qquad\qquad \Big)$$

(d) Fill in the empty spaces in the parentheses to efficiently compute $P(x)$.

# Exercise 6

Determine polynomials $q$ and $r$ so that $\frac{r(x)}{x^2+x-3}$ is a proper fraction and

$$2x^4 + x^3 + 4x - 6 = q(x)(x^2 + x - 3) + r(x).$$

(a) Is the decomposition unique?

(b) What is important about the requirement that the fraction is a proper fraction?

(c) Identify two different decompositions by eliminating the requirement that the fraction is proper.

# Exercise 7

Take $f$ to be the polynomial function that is given by

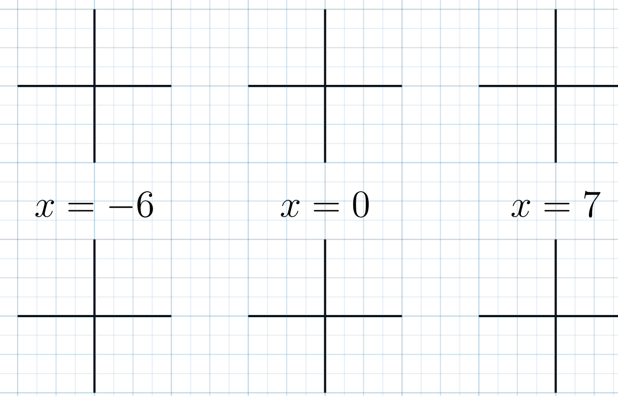
$$f(x) = x(x+6)^4(x-7)^3.$$

(a) What does it mean for $a$ to be a zero of $f$ of order $n$?

(b) List the zeros of $f$ together with their orders.

(c) Determine the shape of $f$ near its zeros. Do not concern yourself with the sign of $f$ near these zeros. Simply sketch all possibilities given only that you know the zeros and their orders.

| zero | order |
| --- | --- |
| | |



# Exercise 8

Take $f$ and $g$ to be the polynomial functions given by

$$f(x) = (x+100)^8(5-2x)^3(4x+1)(x-9)^{24} \quad \textbf{and} \quad g(x) = (x+100)^8(2x-5)^3(4x+1)(x-9)^{25}.$$

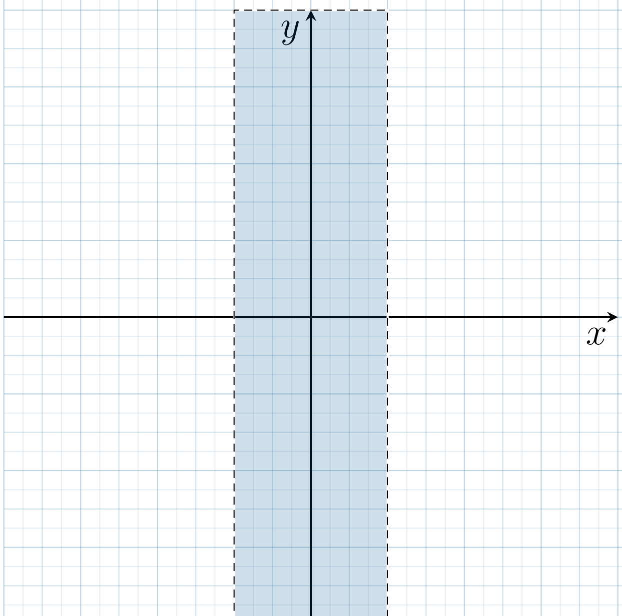
(a) Determine the leading terms of $f$ and $g$.

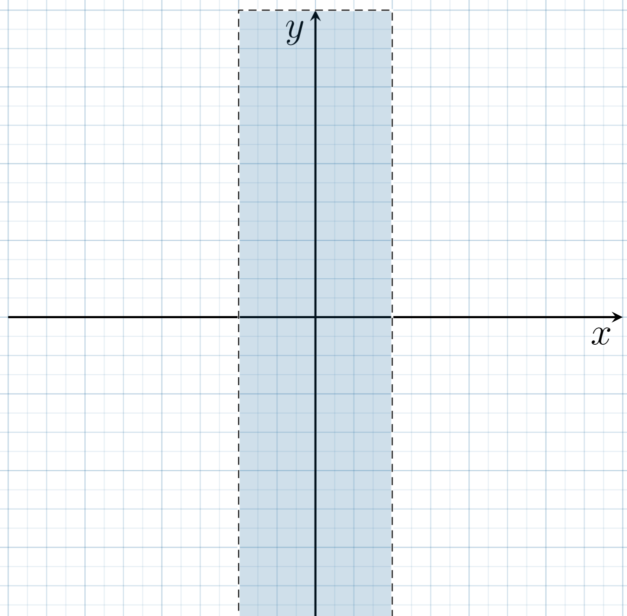
(b) The leading terms define the asymptotic behavior of $f$ and $g$. In plain English, describe what is meant by *asymptotic behavior*.

Take $f$ and $g$ to be the polynomial functions given by

$$f(x) = (x+100)^8(5-2x)^3(4x+1)(x-9)^{24} \quad \text{and} \quad g(x) = (x+100)^8(2x-5)^3(4x+1)(x-9)^{25}.$$

(c) Roughly sketch what $f$ and $g$ each look like far from the origin.

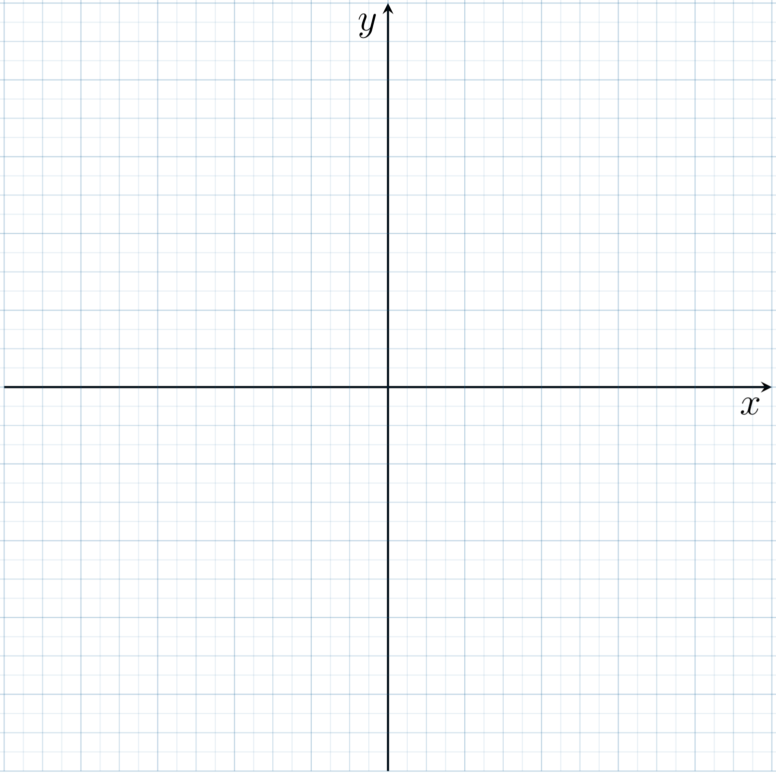


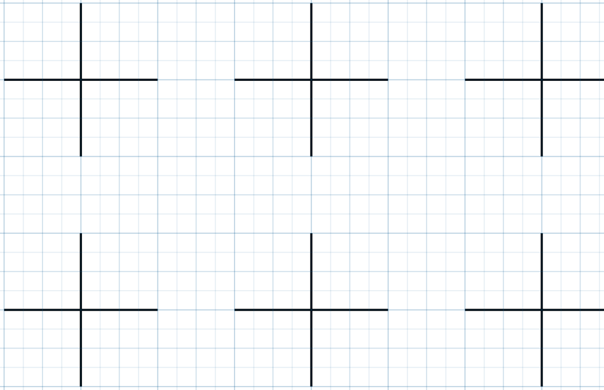



# Exercise 9

Sketch the polynomial $f$ that is given by

$$f(x) = 3(x+6)^2(x+1)(x-7)^2.$$



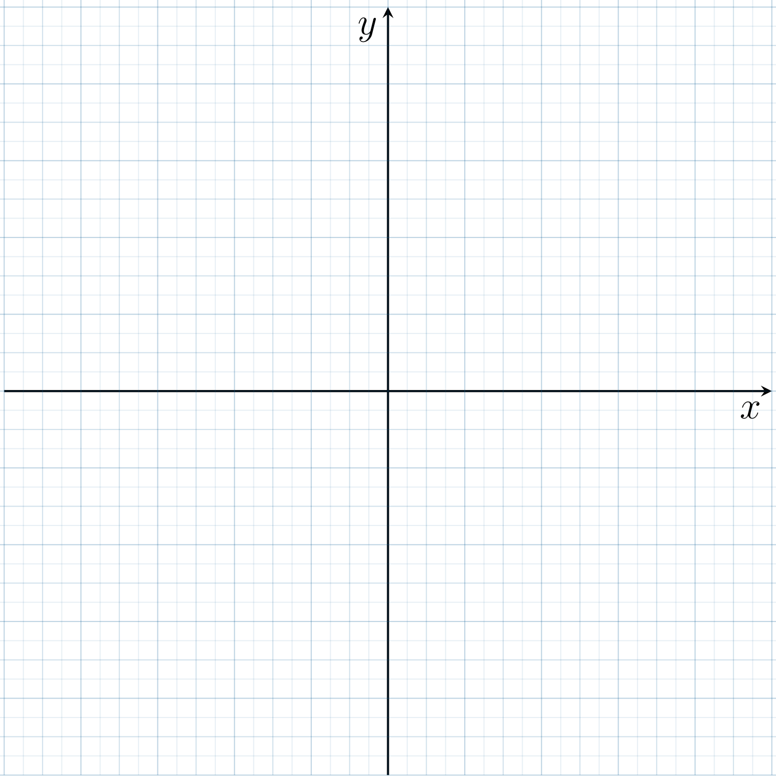



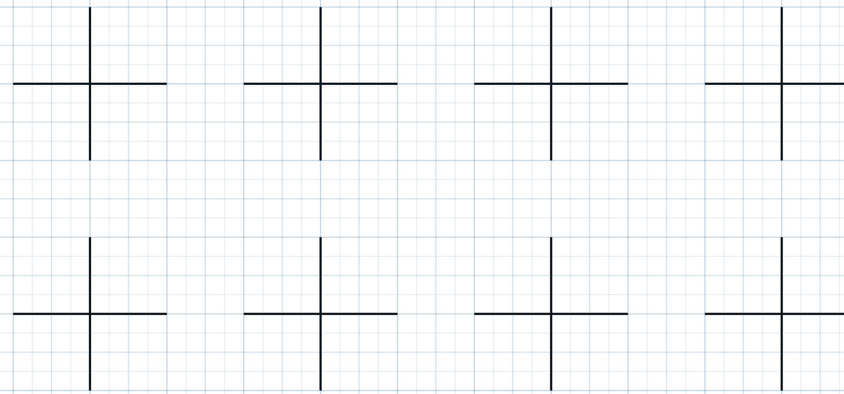
$x = -6$ $x = -1$ $x = 7$

# Exercise 10

Sketch the polynomial $f$ that is given by

$$f(x) = (x+5)^3(x+2)^2(x-3)(x-8)^4.$$



$x = -5$ $\quad$ $x = -2$ $\quad$ $x = 3$ $\quad$ $x = 8$

# Exercise 11

Sketch the polynomial $f$ that is given by

$$f(x) = -(x+5)^3(x+2)^2(x-3)(x-8)^5.$$