1. Use Fermat's theorem to determine all points at which each function f below can potentially attain a local maximum or a local minimum:

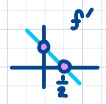
a) $f(x) = -4x^2 + 4x + 3$

$f'(x) = -8x + 4$

Solve $f'(x) = 0$.

$-8x + 4 = 0 \Rightarrow x = \frac{1}{2}$.

Sketch f' to determine sign.



f': $+$ on the left of $\frac{1}{2}$, $-$ on the right

f: $\nearrow$ then $\searrow$

By Fermat's theorem, f has a local maximum at $x_0 = \frac{1}{2}$.

b) $f(x) = x^{\frac{2}{3}} + x$

$f'(x) = \frac{2}{3} x^{-\frac{1}{3}} + 1$

Solve $f'(x) = 0$ and determine where f' is undefined.

i) $\frac{2}{3} x^{-\frac{1}{3}} + 1 = 0$

$\frac{2}{3} x^{-\frac{1}{3}} = -1$

$\frac{2}{3} = -x^{\frac{1}{3}}$ multiply by $x^{\frac{1}{3}}$

$-\left(\frac{2}{3}\right)^3 = x$

$-\frac{8}{27} = x$

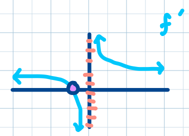
ii) $x^{-\frac{1}{3}} = \frac{1}{x^{\frac{1}{3}}}$

So f' undefined at 0.

Determine sign of f'.

f': $+$, 0 at $-\frac{8}{27}$, $-$, U at 0, $+$

f: $\nearrow$ $\searrow$ $\nearrow$



By Fermat's theorem, f has a local maximum at $-\frac{8}{27}$ and local minimum at 0.

2. Take f to be the function that is given by

$$f(x) = -4x^2 + 4x + 3.$$

Determine the maximum and minimum values that f attains on $[0, 2]$.

Because f is a quadratic polynomial, it is continuous on $[0,2]$. So f attains its maximum and minimum on $[0,2]$.

Determine critical values of f. In 1a we found that the critical value is $\frac{1}{2}$.

Evaluate f on $\{0, \frac{1}{2}, 2\}$ to determine maximum and minimum.

$f(0) = 3$

$f(\frac{1}{2}) = 4$ maximum

$f(2) = -5$ minimum

The maximum of f is at $\frac{1}{2}$ and minimum at 2.

3. Take $f(x) = \frac{1}{4}x^2 + \frac{1}{4}$. This is a differentiable function on the interval $(-1, 2)$ and a continuous function on the interval $[-1, 2]$.

a) Find a value M so that $|f'(x)| \leq M$ for all x in $[-1, 2]$.

$f'(x) = \frac{1}{2}x$. The maximum of abs f' is in the set $\{-1, 0, 2\}$. Note: $f' = 0$ when $x = 0$.

$|f'(-1)| = \frac{1}{2}$, $|f'(0)| = 0$, $|f'(2)| = 1$

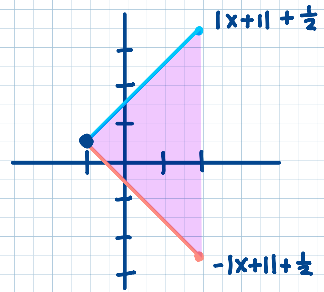
Take $M = 1$. Then $|f'(x)| \leq 1$.

b) Given that $f(-1) = \frac{1}{2}$, find the smallest range of values that is guaranteed to contain $f([-1, 2])$ and sketch the smallest region that you can that is guaranteed to contain f. Then compare it to the graph of f.

By the mean value theorem,

$f(-1) - M|x+1| \leq f(x) \leq f(-1) + M|x+1|$ for all x in $[-1, 2]$

$\frac{1}{2} - |x+1| \leq f(x) \leq \frac{1}{2} + |x+1|$



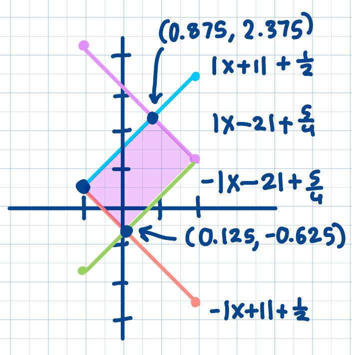
So, $f([-1,2]) \subseteq [-2.5, 3.5]$

c) Given that $f(-1) = \frac{1}{2}$ and $f(2) = \frac{5}{4}$, find the smallest range of values that is guaranteed to contain $f([-1, 2])$ and sketch the smallest region that you can that is guaranteed to contain f. Then compare it to the graph of f.

By the mean value theorem,

$f(-1) - M|x+1| \leq f(x) \leq f(-1) + M|x+1|$ and $f(2) - M|x-2| \leq f(x) \leq f(2) + M|x-2|$ for all x in $[-1, 2]$.

$\frac{1}{2} - |x+1| \leq f(x) \leq \frac{1}{2} - |x+1|$ and $\frac{5}{4} - |x-2| \leq f(x) \leq \frac{5}{4} + |x-2|$



So $f([-1,2]) \subseteq [-0.625, 2.375]$.

4. Take f and g to be differentiable functions on the interval $[1, 9]$ so that $f(2) = 4$, $g(2) = 8$, and that for any x in $(1, 9)$,

$$f'(x) - g'(x) = 0.$$

Determine $g(9) - f(9)$.

Because $f' - g' = 0$ on $(1,9)$, the mean value theorem states that

$f(x) - g(x) = C$ for all x in $[1,9]$ and some constant C.

Because $f(2)=4$ and $g(2)=8$, we have that

$$f(2) - g(2) = C$$
$$4 - 8 = C \quad \Rightarrow f(x) - g(x) = -4 \text{ for all } x \text{ in } [1,9]$$
$$-4 = C$$

So $g(9) - f(9) = 4$.

5. For each function f that is given below, determine the antiderivative of f :

a) $f(x) = 20x^3 + e^x + 2^x + 4\cos(x)$

$$\int f(x)dx = 20\int x^3 dx + \int e^x dx + \int 2^x dx + 4\int \cos(x) dx$$
$$= 20\left(\frac{1}{4}x^4\right) + e^x + \frac{2^x}{\ln(2)} + 4\sin(x) + C$$
$$= 5x^4 + e^x + \frac{2^x}{\ln(2)} + 4\sin(x) + C$$

Can check answer by verifying

$$\left(5x^4 + e^x + \frac{2^x}{\ln(2)} + 4\sin(x) + C\right)' = f(x)$$

b) $f(x) = \frac{3}{x} + \frac{40}{x^2+1} - \ln(2)\sqrt{x} + 2$

$$\int f(x)dx = 3\int \frac{1}{x} dx + 40\int \frac{1}{x^2+1} dx - \ln(2)\int x^{\frac{1}{2}} dx + 2\int 1 dx$$
$$= 3\ln|x| + 40\arctan(x) - \frac{2\ln(2)}{3} x^{\frac{3}{2}} + 2x + C$$

Can check answer by verifying

$$\left(3\ln|x| + 40\arctan(x) - \frac{2\ln(2)}{3} x^{\frac{3}{2}} + 2x + C\right)' = f(x)$$

6. Take f to be the function with the property that

$$\int f(x)\,\mathrm{d}x = \sin(x) + \tan(x) + 2x^3 + C.$$

Identify the function f.

$F(x) = \sin(x) + \tan(x) + 2x^3$ is an antiderivative of f so,

$$F'(x) = f(x).$$

Hence

$$f(x) = (\sin(x) + \tan(x) + 2x^3)'$$
$$= \cos(x) + \sec^2(x) + 6x^2.$$

7. Calculate the following:

a) $\displaystyle\int_1^2 x^5\,\mathrm{d}x$

Since $\int x^5 dx = \frac{1}{6}x^6 + C$, we have by the Fundamental Theorem of Calculus that

$$\int_1^2 x^5 dx = \left(\tfrac{1}{6}x^6\right)\Big|_1^2$$
$$= \frac{2^6}{6} - \frac{1^6}{6}$$
$$= \frac{21}{2}.$$

b) $\displaystyle\int_0^1 (20x^3 + \mathrm{e}^x + 2^x + 4\cos(x))\,\mathrm{d}x$

Since $F(x) = 5x^4 + e^x + \frac{2^x}{\ln(2)} + 4\sin(x)$ is an antiderivative of $f(x) = 20x^3 + e^x + 2^x + 4\cos(x)$, we have by the Fundamental Theorem of Calculus that

$$\int_0^1 (20x^3 + e^x + 2^x + 4\cos(x))dx = (F(x))\Big|_0^1$$
$$= F(1) - F(0)$$
$$= 4 + e + \frac{1}{\ln(2)} + 4\sin(1).$$

c) Derivative of F where $F(x) = \displaystyle\int_2^x \sin(t^2)\,\mathrm{d}t$.

Because $f(t) = \sin(t^2)$ is continuous on $[2, x]$ for all $x > 2$, we have by the Fundamental Theorem of Calculus that F is differentiable on $(2, \infty)$ and

$$F'(x) = f(x) = \sin(x^2).$$

d) Derivative of F where $F(x) = \displaystyle\int_2^{4x+2} \sin(t^2)\,\mathrm{d}t$.

Take $G(x) = \int_2^x \sin(t^2)dt$. Then,

$$F(x) = (G \circ (4\mathrm{pow}_1 + 2))(x), \text{ where } 4x+2>2$$
$$x>0$$

So by chain rule, FTC, and 7c,

$$F'(x) = G'(4x+2) \cdot (4x+2)'$$
$$= \sin((4x+2)^2) \cdot 4$$

By 7c $G'(x) = \sin(x^2)$

$$= 4\sin((4x+2)^2) \quad \text{for } x > 0.$$

8. Determine the following:

a) $\displaystyle\int 4x^3 \sin(x^4)\,\mathrm{d}x$

The integrand looks like this

$$4x^3\sin(x^4) = -(x^4)'\cos'(x^4) = -(\cos(x^4))'$$

So by reverse chain rule

$$\int 4x^3\sin(x^4)dx = -\int(\cos(x^4))'dx$$

$$= -\cos(x^4) + C.$$

Alternatively,

$$\int 4x^3\sin(x^4)dx = \int \sin(x^4)\,4x^3dx$$

$u = x^4$

$du = 4x^3dx$

$$= \int \sin(u)\,du$$

$$= -\cos(u) + C$$

$$= -\cos(x^4) + C.$$

b) $\displaystyle\int (40x^3 + x)\sqrt{20x^4 + x^2 + 1}\,\mathrm{d}x$

The integrand looks like this

$$(40x^3+x)\sqrt{20x^4+x^2+1} = \frac{1}{2}(80x^3+2x)\sqrt{20x^4+x^2+1}$$

$$= \frac{1}{2}(20x^4+x^2+1)'\,\mathrm{pow}_{\frac{1}{2}}(20x^4+x^2+1)$$

$$= \frac{1}{2}(20x^4+x^2+1)'\,\frac{2}{3}\,\mathrm{pow}'_{\frac{3}{2}}(20x^4+x^3+1)$$

$$= \frac{1}{3}\left(\mathrm{pow}_{\frac{3}{2}}(20x^4+x^2+1)\right)'$$

So by reverse chain rule

$$\int(40x^3+x)\sqrt{20x^4+x^2+1}\,dx = \frac{1}{3}\int\left(\mathrm{pow}_{\frac{3}{2}}(20x^4+x^2+1)\right)'dx$$

$$= \frac{1}{3}(20x^4+x^2+1)^{\frac{3}{2}} + C.$$

Alternatively

$$\int(40x^3+x)\sqrt{20x^4+x^2+1}\,dx = \frac{1}{2}\int\sqrt{u}\,du$$

$u = 20x^4+x^2+1$

$$= \frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C$$

$du = (80x^3+2x)dx$

$\frac{1}{2}du = (40x^3+2)dx$

$$= \frac{1}{3}(20x^4+x^2+1) + C.$$

c) $\displaystyle\int \frac{\mathrm{e}^x + \ln(2)2^x}{\mathrm{e}^x + 2^x}\,\mathrm{d}x$

The integrand looks like this

$$e^x+\ln(2)2^x\cdot\frac{1}{e^x+2^x} = (e^x+2^x)'\cdot \mathrm{Recip}(e^x+2^x)$$

$$= (\ln|e^x+2^x|)'$$

So by reverse chain rule

$$\int\frac{e^x+\ln(2)2^x}{e^x+2}dx = \int(\ln|e^x+2^x|)'dx$$

$$= \ln|e^x+2^x| + C.$$

Alternatively,

$$\int\frac{e^x+\ln(2)2^x}{e^x+2}dx = \int\frac{1}{u}\,du$$

$u = e^x+2^x$

$$= \ln|u| + C$$

$du = (e^x+\ln(2)2^x)dx$

$$= \ln|e^x+2^x| + C.$$

9. Use the fact that

$$(x\cos(x))' = \cos(x) - x\sin(x)$$

to determine

$$\int x\sin(x)\,\mathrm{d}x.$$

$(x\cos(x))' = \cos(x) - x\sin(x)$ implies $x\sin(x) = \cos(x) - (x\cos(x))'$

So

$$\int x\sin(x)\,dx = \int \cos(x)\,dx - \int (x\cos(x))'\,dx$$

$$= \sin(x) - x\cos(x) + C.$$

10. The function $v(t)$ given by $v(t) = \langle 5t+2, 5\rangle$ is the velocity of a path c at time t. Given that $c(2) = (4,5)$, reconstruct the path c and simulate the motion of the particle together with the particle's velocity vector.

We have $c'(t) = v(t)$ so

$$c(t) = \left(\int (5t+2)\,dt, \int 5\,dt\right)$$

$$= \left(\tfrac{5}{2}t^2 + 2t + x_0, 5t + y_0\right) \text{ for some } x_0 \text{ and } y_0 \text{ in } \mathbb{R}.$$

Since $c(2) = (4,5)$,

$$c(2) = \left(\tfrac{5}{2}(2)^2 + 2(2) + x_0, 5(2) + y_0\right)$$

$$(4,5) = (14 + x_0, 10 + y_0) \Rightarrow \begin{cases} 4 = 14 + x_0 \\ 5 = 10 + y_0 \end{cases} \Rightarrow \begin{matrix} x_0 = -10 \\ y_0 = -5. \end{matrix}$$

Thus

$$c(t) = \left(\tfrac{5}{2}t^2 + 2t - 10, 5t - 5\right).$$

11 Verify that the following are of indeterminate forms and use L'Hopital's rule to determine the given limits:

a) $\lim\limits_{x\to 5} \dfrac{x^3-24x-5}{x-5}$

$\lim\limits_{x\to 5}(x^3-24x-5)=0$ and $\lim\limits_{x\to 5}(x-5)=0$.

This is an indeterminant form.

By L'Hopital's rule

$$\lim_{x\to 5}\frac{x^3-24x-5}{x-5}=\lim_{x\to 5}\frac{(x^3-24x-5)'}{(x-5)'} \quad \text{L'H Rule}$$
$$=\lim_{x\to 5}(3x^2-24)$$
$$=3(5)^2-24$$
$$=51.$$

b) $\lim\limits_{x\to\infty}\left(\sqrt{4x^2+3x+1}-2x\right)$

$\lim\limits_{x\to\infty}\sqrt{4x^2+3x+1}=\infty$ and $\lim\limits_{x\to\infty}2x=\infty$.

This is an indeterminant form.

By L'Hopital's rule

$$\lim_{x\to\infty}\left(\sqrt{4x^2+3x+1}-2x\right)=\lim_{x\to\infty}\left(\sqrt{4x^2\left(1+\tfrac{3}{4x}+\tfrac{4}{x^2}\right)}-2x\right)$$
$$=\lim_{x\to\infty}\left(2x\sqrt{1+\tfrac{3}{4x}+\tfrac{4}{x^2}}-2x\right) \quad \sqrt{4x^2}=2x,\ x>0$$
$$=\lim_{x\to\infty}\frac{\sqrt{1+\frac{3}{4x}+\frac{4}{x^2}}-1}{\left(\frac{1}{2x}\right)} \quad \text{“}\tfrac{0}{0}\text{”}$$
$$=\lim_{x\to\infty}\frac{\left(\sqrt{1+\frac{3}{4x}+\frac{4}{x^2}}-1\right)'}{\left(\frac{1}{2x}\right)'} \quad \text{L'H Rule}$$
$$=\lim_{x\to\infty}\frac{\left(\frac{1}{2\sqrt{1+\frac{3}{4x}+\frac{4}{x^2}}}\right)\left(-\frac{3}{4x^2}-\frac{8}{x^3}\right)}{\left(-\frac{1}{2x^2}\right)}$$
$$=\lim_{x\to\infty}\left(\frac{1}{2\sqrt{1+\frac{3}{4x}+\frac{4}{x^2}}}\right)\left(\frac{3}{2}+\frac{16}{x}\right)$$
$$=\frac{1}{2\sqrt{1+0+0}}\cdot\left(\frac{3}{2}+0\right)$$
$$=\frac{3}{4}.$$

c) $\lim\limits_{x\to\infty}\dfrac{x^3}{e^x}$

$\lim\limits_{x\to\infty}x^3=\infty$ and $\lim\limits_{x\to\infty}e^x=\infty$.

This is an indeterminant form.

By L'Hoptial's rule

$$\lim_{x\to\infty}\frac{x^3}{e^x}=\lim_{x\to\infty}\frac{(x^3)'}{(e^x)'} \quad \text{L'H Rule}$$
$$=\lim_{x\to\infty}\frac{3x^2}{e^x} \quad \tfrac{\infty}{\infty}$$
$$=\lim_{x\to\infty}\frac{(3x^2)'}{(e^x)'} \quad \text{L'H Rule}$$
$$=\lim_{x\to\infty}\frac{6x}{e^x} \quad \tfrac{\infty}{\infty}$$
$$=\lim_{x\to\infty}\frac{6}{e^x} \quad \text{L'H Rule}$$
$$=0.$$