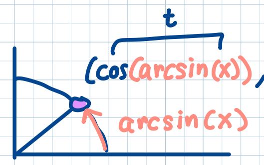
1. Show the derivative of $f(x) = \arcsin(x)$ is $f'(x) = \dfrac{1}{\sqrt{1-x^2}}$.

For x in $[-\frac{\pi}{2}, \frac{\pi}{2}]$, the inverse of sin is arcsin.

So

$$(\arcsin)'(x) = \frac{1}{\sin'(\arcsin(x))} = \frac{1}{\cos(\arcsin(x))} = \frac{1}{\sqrt{1-x^2}} \quad \text{for } x \text{ in } (-1,1)$$



$x^2 + t^2 = 1$

$t = \sqrt{1-x^2}$

$t \geq 0$ when x in $[-\frac{\pi}{2}, \frac{\pi}{2}]$

2. Compute the derivative of the function $f(x) = \arcsin\left(\ln\left(\frac{1}{\sqrt{1-x^2}}\right)\right)$.

$$f = \underbrace{\arcsin \circ \ln \circ}_{\text{chain}} \left(\frac{1}{\underbrace{\mathrm{pow}_{\frac{1}{2}}(1 - \mathrm{pow}_2)}_{\text{quotient or reciprocal}}}\right)$$

So

$$f'(x) = \arcsin'\left(\ln\left(\frac{1}{\sqrt{1-x^2}}\right)\right) \cdot \ln'\left(\frac{1}{\sqrt{1-x^2}}\right) \cdot \left(\frac{1}{\sqrt{1-x^2}}\right)' \quad \text{Chain}$$

$$= \frac{1}{\sqrt{1-\left(\ln\left(\frac{1}{\sqrt{1-x^2}}\right)\right)^2}} \cdot \frac{1}{\left(\frac{1}{\sqrt{1-x^2}}\right)} \cdot \left(\frac{\left(\frac{x}{\sqrt{1-x^2}}\right)}{1-x^2}\right)$$

$$= \frac{1}{\sqrt{1-\left(\ln\left(\frac{1}{\sqrt{1-x^2}}\right)\right)^2}} \cdot \left(\frac{x}{\sqrt{1-x^2}}\right)$$

3. Take $f(x) = \left(\frac{(\cos(x)-2x)(\exp_2(x)+4)}{\sin(x)+4}\right)^3$. Determine f' by using logarithm differentiation.

$$\ln(f(x)) = 3\ln(\underbrace{\cos(x) - 2x}_{\text{chain}}) + 3\ln(\underbrace{\exp_2(x)+4}_{\text{chain}}) - 3\ln(\underbrace{\sin(x)+4}_{\text{chain}})$$

So

$$(\ln(f(x)))' = -\frac{3\sin(x)+6}{\cos(x)-2x} + \frac{3\ln(2)2^x}{2^x+4} - \frac{3\cos(x)}{\sin(x)+4}.$$

Hence

$$f'(x) = (\ln(f(x)))' f(x)$$

$$= \left(-\frac{3\sin(x)+6}{\cos(x)-2x} + \frac{3\ln(2)2^x}{2^x+4} - \frac{3\cos(x)}{\sin(x)+4}\right)\left(\frac{(\cos(x)-2x)(2^x+4)}{\sin(x)+4}\right)^3.$$

4. Take $f(x) = (x+3)^{\sin(x)}$. Determine $f'(x)$.

Note

$$(x+3)^{\sin(x)} = \exp(\ln((x+3)^{\sin(x)})) = \exp(\underbrace{\overbrace{\sin(x)\ln(x+3)}^{\text{product}}}_{\text{chain}}).$$

So

$$f'(x) = \exp'(\sin(x)\ln(x+3)) \cdot (\sin(x)\ln(x+3))' \quad \text{chain}$$

$$= \exp(\sin(x)\ln(x+3))\left(\cos(x)\ln(x+3) + \frac{\sin(x)}{x+3}\right) \quad \text{product}$$

$$= (x+3)^{\sin(x)}\left(\cos(x)\ln(x+3) + \frac{\sin(x)}{x+3}\right)$$

5. Take f to be the function that is given by $f(x,y) = 5x^2y^4 + 4xy + \csc(-8x+3y^2)$. Determine $f_x(1,2)$ and $f_y(1,2)$.

Note that $f_x(x,y) = 5y^4(x^2)' + 4y(x)' + \csc'(-8x+3y^2)(-8x)'$

$$= 10xy^4 + 4y + 8\cot(-8x+3y^2)\csc(-8x+3y^2)$$

So $f_x(1,2) = 168 + 8\cot(4)\csc(4)$.

Note that $f_y(x,y) = 5x^2(y^4)' + 4x(y)' + \csc'(-8x+3y^2)(3y^2)'$

$$= 20x^2y^3 + 4x - 6y\cot(-8x+3y^2)\csc(-8x+3y^2)$$

So $f_y(1,2) = 164 - 12\cot(4)\csc(4)$.

6. Assume that y is defined implicitly by the equation $y^2x + \arctan(y+2) = xy$. Calculate $\frac{dy}{dx}$.

Here y is a function of x. So

$$\frac{d}{dx}\Big(\underbrace{\overbrace{y^2}^{\text{chain}}x}_{\text{Product}} + \overbrace{\arctan(y+2)}^{\text{chain}}\Big) = \frac{d}{dx}(\underbrace{xy}_{\text{product}})$$

$$2y\frac{dy}{dx}x + y^2 + \frac{1}{(y+2)^2+1}\frac{dy}{dx} = y + x\frac{dy}{dx}$$

$$\frac{dy}{dx}\left(2y - x + \frac{1}{(y+2)^2+1}\right) = y - y^2 \quad \text{rearrange}$$

$$\frac{dy}{dx} = \frac{y - y^2}{2y - x + \frac{1}{(y+2)^2+1}}$$

7. The equation $x^4y - xy^8 = -899934390$ implicitly defines y as a function of x in an open rectangle around the point $(9,10)$. Determine an equation for the line that is tangent to the solution set to the equation at the point $(9,10)$.

Here y is a function of x. So

$$\frac{d}{dx}(\underbrace{x^4y}_{\text{Product}} - \underbrace{xy^8}_{\text{Product}}) = \frac{d}{dx}(-899934390)$$

$$4x^3y + x^4\frac{dy}{dx} - y^8 - 8xy^7\frac{dy}{dx} = 0$$

$$(x^4 - 8xy^7)\frac{dy}{dx} = y^8 - 4x^3y \quad \text{rearrange}$$

$$\frac{dy}{dx} = \frac{y^8 - 4x^3y}{x^4 - 8xy^7}.$$

Evaluate $\frac{dy}{dx}$ at $(9,10)$ to obtain

$$\left.\frac{dy}{dx}\right|_{(9,10)} = \frac{10^8 - 4(9)^3(10)}{9^4 - 8(9)(10)^7} = -\frac{99999640}{719993439}.$$

The line tangent to the equation at the point $(9,10)$ is

$$y = -\frac{99999640}{719993439}(x-9) + 10$$

8. A spherical balloon is being inflated with air at a rate so its volume is increasing at a rate of 150 centimeters cubed per second. Determine the rate at which the surface area is changing when the diameter of the balloon is 65 centimeters.

Take r to be radius at time t and A to be surface area at time t. Then

$$A(t) = 4\pi(r(t))^2 \quad \text{so} \quad \frac{dA}{dt} = 8\pi r(t)\frac{dr}{dt}.$$

Need $\left.\frac{dr}{dt}\right|_{r=65}$.

Take V to be the volume of spherical balloon at time t. Then

$$V(t) = \frac{4}{3}\pi r^3(t) \quad \text{so} \quad \frac{dV}{dt} = 4\pi r^2(t)\frac{dr}{dt} \quad \text{or} \quad \frac{dr}{dt} = \frac{1}{4\pi r^2(t)}\frac{dV}{dt}.$$

Since $\left.\frac{dV}{dt}\right|_{r=65} = 150\,\frac{cm^3}{s}$, we have

$$\left.\frac{dr}{dt}\right|_{r=65} = \frac{1}{4\pi(65)^2 cm^2}\,150\,\frac{cm^3}{s} = \frac{3}{338\pi}\,\frac{cm}{s}.$$

Hence

$$\left.\frac{dA}{dt}\right|_{r=65} = 8\pi(65)\cdot\left.\frac{dr}{dt}\right|_{r=65} = 8\pi(65)\,cm \cdot \frac{3}{338\pi}\,\frac{cm}{s} = \frac{60}{13}\,\frac{cm}{s}.$$