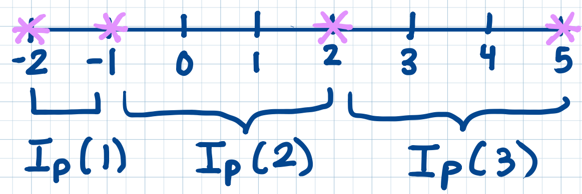
1. A partition P of the set $[-2, 5]$ has domain $\{0, 1, 2, 3\}$ and $P = (-2, -1, 2, 5)$. Determine the mesh of P.

$P(0) = -2$, $P(1) = -1$, $P(2) = 2$, $P(3) = 5$.

$P(1) - P(0) = 1$

$P(2) - P(1) = 3$

$P(3) - P(2) = 2$

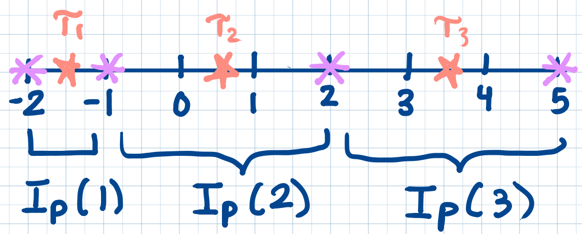


So $\|P\| = \max\{1, 3, 2\} = 3$.

2. Identify a midpoint tagging τ of this partition P of $[-2, 5]$ that has three intervals and $P = (-2, -1, 2, 5)$.

$P(0) = -2$, $P(1) = -1$, $P(2) = 2$, $P(3) = 5$.

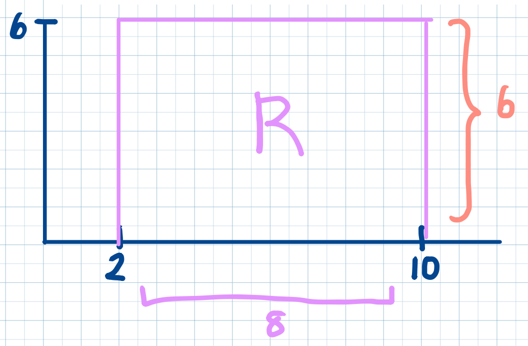
$\tau_1 = \frac{-2-1}{2} = -\frac{3}{2}$, $\tau_2 = \frac{-1+2}{2} = \frac{1}{2}$, $\tau_3 = \frac{2+5}{2} = \frac{7}{2}$



$\tau = \left(-\frac{3}{2}, \frac{1}{2}, \frac{7}{2}\right)$

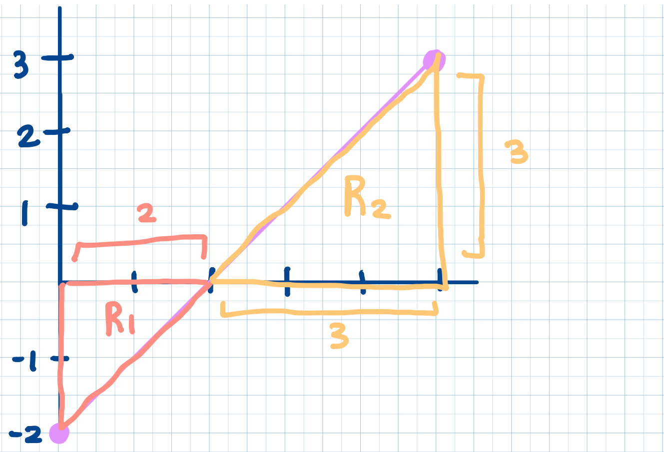
3. Calculate the area of each region R formed by bounding the following.

a) $y = 6$, $x = 2$, $x = 10$ and the x-axis



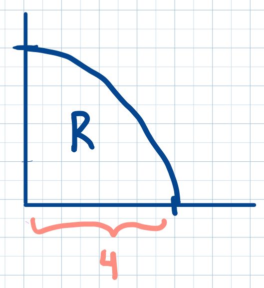
$$\text{Area}(R) = 8 \cdot 6 = 48$$

b) $y = x - 2$, $x = 0$, $x = 5$ and the x-axis



$$\begin{aligned}\text{Area}(R) &= \text{Area}(R_1) + \text{Area}(R_2)\\ &= -\tfrac{1}{2}(2)(2) + \tfrac{1}{2}(3)(3)\\ &= -2 + \tfrac{9}{2}\\ &= \tfrac{5}{2}\end{aligned}$$

c) $y = \sqrt{16 - x^2}$, $x = 0$, $x = 4$ and the x-axis

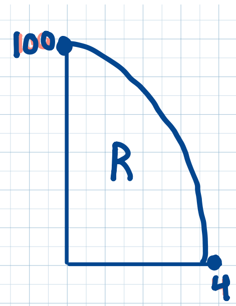


$$y = \sqrt{16 - x^2}$$
$$y^2 = 16 - x^2$$
$$\underbrace{x^2 + y^2 = 16}_{C_4(0,0)}$$

circle
center $(0,0)$
radius 4

$$\begin{aligned}\text{Area}(R) &= \tfrac{1}{4}\,\text{Area}(C_4(0,0))\\ &= \tfrac{1}{4}\pi(4)^2\\ &= 4\pi\end{aligned}$$

d) $y = 25\sqrt{16 - x^2}$, on $[0, 4]$ and the x-axis.



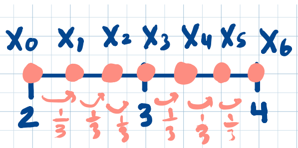
$$y = 25\sqrt{16 - x^2}$$
$$\frac{y}{25} = \sqrt{16 - x^2}$$
$$\frac{y^2}{(25)^2} = 16 - x^2$$
$$x^2 + \frac{y^2}{(25)^2} = 16$$
$$\underbrace{\frac{x^2}{4^2} + \frac{y^2}{(4 \cdot 25)^2} = 1}_{\mathcal{E}_{4,100}(0,0)}$$

Ellipse center $(0,0)$
x-scaling 4
y-scaling 100

$$\begin{aligned}\text{Area}(R) &= \tfrac{1}{4}\,\text{Area}(\mathcal{E}_{4,100}(0,0))\\ &= \tfrac{1}{4}\pi\, 4 \cdot 100\\ &= 100\pi.\end{aligned}$$

4. Take R to be the region formed by bounding $y = 4x^2 + 1$, x-axis, $x = 2$, and $x = 4$. Approximate R by using six left-endpoint rectangles, six right-endpoint rectangles, and six midpoint rectangles.

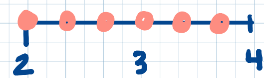
$$\Delta x = \frac{4-2}{6} = \frac{1}{3}$$



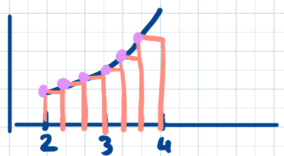
$x_0 = 2$, $x_1 = \frac{7}{3}$, $x_2 = \frac{8}{3}$, $x_3 = 3$, $x_4 = \frac{10}{3}$, $x_5 = \frac{11}{3}$, $x_6 = 4$

Left endpoint \ Left tagging

$T_1 = 2$, $T_2 = \frac{7}{3}$, $T_3 = \frac{8}{3}$, $T_4 = 3$, $T_5 = \frac{10}{3}$, $T_6 = \frac{11}{3}$

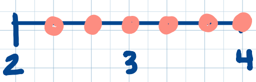


$$f(2)\cdot\tfrac{1}{3} + f\left(\tfrac{7}{2}\right)\cdot\tfrac{1}{3} + f\left(\tfrac{8}{3}\right)\cdot\tfrac{1}{3} + f(3)\cdot\tfrac{1}{3} + f\left(\tfrac{10}{3}\right)\cdot\tfrac{1}{3} + f\left(\tfrac{11}{3}\right)\cdot\tfrac{1}{3} = \frac{1858}{27}$$

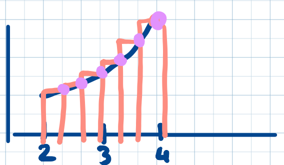


Right endpoint \ Right tagging

$T_1 = \frac{7}{3}$, $T_2 = \frac{8}{3}$, $T_3 = 3$, $T_4 = \frac{10}{3}$, $T_5 = \frac{11}{3}$, $T_6 = 4$

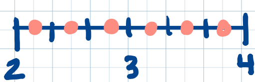


$$f\left(\tfrac{7}{3}\right)\cdot\tfrac{1}{3} + f\left(\tfrac{8}{3}\right)\cdot\tfrac{1}{3} + f(3)\cdot\tfrac{1}{3} + f\left(\tfrac{10}{3}\right)\cdot\tfrac{1}{3} + f\left(\tfrac{11}{3}\right)\cdot\tfrac{1}{3} + f(4)\cdot\tfrac{1}{3} = \frac{2290}{27}$$



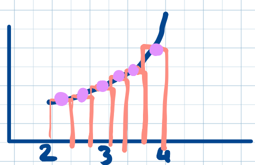
Midpoint tagging

$T_1 = \dfrac{2+\frac{7}{3}}{2} = \dfrac{13}{6}$, $T_2 = \dfrac{\frac{7}{3}+\frac{8}{3}}{2} = \dfrac{15}{6}$, $T_3 = \dfrac{\frac{8}{3}+3}{2} = \dfrac{17}{6}$,



$T_4 = \dfrac{3+\frac{10}{3}}{2} = \dfrac{19}{6}$, $T_5 = \dfrac{\frac{10}{3}+\frac{11}{3}}{2} = \dfrac{21}{6}$, $T_6 = \dfrac{\frac{11}{3}+4}{2} = \dfrac{23}{6}$

$$f\left(\tfrac{13}{6}\right)\cdot\tfrac{1}{3} + f\left(\tfrac{15}{6}\right)\cdot\tfrac{1}{3} + f\left(\tfrac{17}{6}\right)\cdot\tfrac{1}{3} + f\left(\tfrac{19}{6}\right)\cdot\tfrac{1}{3} + f\left(\tfrac{21}{6}\right)\cdot\tfrac{1}{3} + f\left(\tfrac{23}{6}\right)\cdot\tfrac{1}{3} = \frac{2068}{27}$$

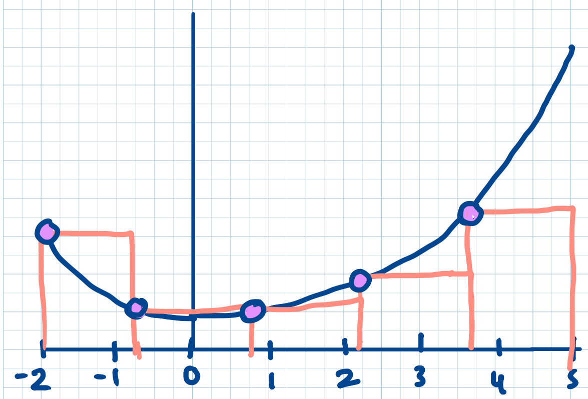


5. Take f to be the function given by $f(x) = 4x^2 + 1$ and the interval I to be the interval $I = [-2, 5]$. Determine an even partition P of I with five intervals and a left tagging τ for P and calculate the quantity $\mathcal{R}(f, P, \tau)$.

$$\Delta x = \frac{5+2}{5} = \frac{7}{5} \Rightarrow P = \left(-2, -\frac{3}{5}, \frac{4}{5}, \frac{11}{5}, \frac{18}{5}, 5\right)$$

$$\tau_1 = -2, \quad \tau_2 = -\frac{3}{5}, \quad \tau_3 = \frac{4}{5}, \quad \tau_4 = \frac{11}{5}, \quad \tau_5 = \frac{18}{5}$$

$$\mathcal{R}(f, P, \tau) = f(-2)\cdot\frac{7}{5} + f\left(-\frac{3}{5}\right)\cdot\frac{7}{5} + f\left(\frac{4}{5}\right)\cdot\frac{7}{5} + f\left(\frac{11}{5}\right)\cdot\frac{7}{5} + f\left(\frac{18}{5}\right)\cdot\frac{7}{5} = \frac{3367}{25}$$



6. Calculate using Riemann sums the area of the region R formed by bounding $y = 4x^2 + 1$, $x = 2$, $x = 4$, and the x-axis.

$\Delta x = \frac{4-2}{n} = \frac{2}{n}$, $x_i = 2 + \frac{2}{n}i \quad i = 1, 2, \ldots, n$

$$\sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n}\left(4\left(2+\frac{2}{n}i\right)^2+1\right)\frac{2}{n}$$

$$= \sum_{i=1}^{n}\left(4\left(4+\frac{8}{n}i+\frac{4}{n^2}i^2\right)+1\right)\frac{2}{n}$$

$$= \sum_{i=1}^{n}\left(17+\frac{32}{n}i+\frac{16}{n^2}i^2\right)\cdot\frac{2}{n}$$

$$= \sum_{i=1}^{n}\left(\frac{34}{n}+\frac{64i}{n^2}+\frac{32}{n^3}i^2\right)$$

$$= \frac{34}{n}\sum_{i=1}^{n}1 + \frac{64}{n^2}\sum_{i=1}^{n}i + \frac{32}{n^3}\sum_{i=1}^{n}i^2$$

$$= \frac{34}{n}\cdot n + \frac{64}{n^2}\cdot\frac{n(n+1)}{2} + \frac{32}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$$

$$= 34 + \frac{32(n+1)}{n} + \frac{16}{3}\frac{(n+1)(2n+1)}{n^2}.$$

So

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty}\left(34+\frac{32(n+1)}{n}+\frac{16}{3}\frac{(n+1)(2n+1)}{n^2}\right)$$

$$= 34 + 32\cdot 1 + \frac{16}{3}\cdot 2$$

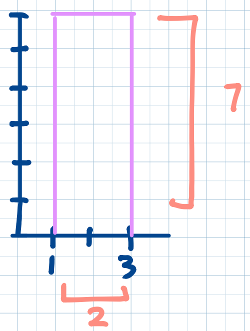
$$= \frac{230}{4}.$$

Thus

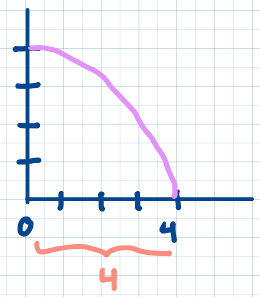
$$\text{Area}(R) = \frac{230}{4}$$

7. Calculate the following by interpreting the definite integral as signed area.

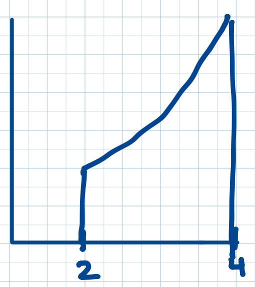
a) $\int_1^3 7\,\mathrm{d}x$ = $7 \cdot 2 = 14$



b) $\int_0^4 \sqrt{16-x^2}\,\mathrm{d}x$ = $\frac{1}{4}\pi(4)^2 = 4\pi$



c) $\int_2^4 (4x^2+1)\,\mathrm{d}x$ = $\frac{230}{4}$ See Question 6



8. Take f and g to be integrable functions with

$$\int_2^5 f(x)\,\mathrm{d}x = -2 \quad \text{and} \quad \int_2^5 g(x)\,\mathrm{d}x = 6.$$

Calculate the following:

$$\int_2^5 (2f(x) + g(x) + 4)\,\mathrm{d}x.$$

$$\int_2^5 (2f(x)+g(x)+4)dx = 2\int_2^5 f(x)\,dx + \int_2^5 g(x)\,dx + \int_2^5 4\,dx$$

$$= 2(-2) + 6 + 4(5-2)$$

$$= 14.$$

9. Take f to be an integrable function with

$$2 \leq f(x) \leq 7$$

on the interval $[-2, 5]$. Calculate an upper and lower bound for the following:

a) $\underbrace{\int_0^3 f(x)\,\mathrm{d}x}_{I}$

$$2(3-0) \leq I \leq 7(3-0)$$

$$6 \leq I \leq 21$$

b) $\underbrace{\int_{-2}^5 f(x)\,\mathrm{d}x}_{J}$

$$2(5+2) \leq J \leq 7(5+2)$$

$$14 \leq J \leq 49$$

c) $\underbrace{\int_{-2}^5 2f(x)\,\mathrm{d}x}_{K}$

$2 \leq f(x) \leq 7$ on $[-2,5]$

$\Downarrow$

$4 \leq 2f(x) \leq 14$ on $[-2,5]$

$$4(5+2) \leq K \leq 14(5+2)$$

$$28 \leq K \leq 98$$

d) $\int_{-2}^5 \underbrace{(f(x) + 3)\,\mathrm{d}x}_{L}$

$2 \leq f(x) \leq 7$ on $[-2,5]$

$\Downarrow$

$$5 \leq f(x)+3 \leq 10$$

$$5(5+2) \leq L \leq 10(5+2)$$

$$35 \leq L \leq 70$$

10. Calculate the following:

a) $\displaystyle\int_{-2}^{3} x^4 \, \mathrm{d}x$

$[-2,0]\cup[0,3]=[-2,3]$

So,

$$\int_{-2}^{3} x^4 dx = \int_{-2}^{0} x^4 dx + \int_{0}^{3} x^4 dx$$

$$= \int_{0}^{2} x^4 dx + \int_{0}^{3} x^4 dx$$

$$= \frac{1}{5}(2)^5 + \frac{1}{5}(3)^5$$

Pow$_4$ even so

$$\int_{-2}^{0} x^4 dx = \int_{0}^{2} x^4 dx.$$

$$\int_{0}^{b} x^4 dx = \frac{1}{5}(b)^5.$$

b) $\displaystyle\int_{-2}^{3} x^{\frac{1}{3}} \, \mathrm{d}x$

$[-2,2]\cup[2,3]=[-2,3]$

So

$$\int_{-2}^{3} x^{\frac{1}{3}} dx = \int_{-2}^{2} x^{\frac{1}{3}} dx + \int_{2}^{3} x^{\frac{1}{3}} dx$$

$$= \int_{2}^{3} x^{\frac{1}{3}} dx$$

since pow$_{\frac{1}{3}}$ odd so

$$\int_{-2}^{2} x^{\frac{1}{3}} dx = 0.$$

$[0,2]\cup[2,3]=[0,3]$

So

$$\int_{0}^{2} x^{\frac{1}{3}} dx + \int_{2}^{3} x^{\frac{1}{3}} dx = \int_{0}^{3} x^{\frac{1}{3}} dx.$$

Thus,

$$\int_{2}^{3} x^{\frac{1}{3}} dx = \int_{0}^{3} x^{\frac{1}{3}} dx - \int_{0}^{2} x^{\frac{1}{3}} dx$$

$$= \frac{3}{4}(3)^{\frac{4}{3}} - \frac{3}{4}(2)^{\frac{4}{3}}$$

$$\int_{0}^{b} x^{\frac{1}{3}} dx = \frac{3}{4}(b)^{\frac{4}{3}}$$