1. For each function f and each x_0 that is given below, determine the difference quotient $\frac{1}{h}\Delta_{x_0}f(h)$ for each real number h.

a) $f(x) = -2x^2 + x + 3$, $x_0 = 2$

$$\Delta_2 f(h) = f(2+h) - f(2)$$
$$= -2(2+h)^2 + (2+h) + 3 + 3$$
$$= -2(4+4h+h^2) + (2+h) + 6$$
$$= -8 - 8h - 2h^2 + 2 + h + 6$$
$$= -7h - 2h^2$$
$$= h(-7-2h)$$

So

$$\frac{1}{h}\Delta_2 f(h) = \frac{1}{h}\cdot h(-7-2h)$$
$$= -7 - 2h.$$

b) $f(x) = \frac{x^2-3}{x}$, $x_0 = 5$

$$\Delta_5 f(h) = f(5+h) - f(5)$$
$$= \frac{(5+h)^2-3}{5+h} - \frac{22}{5}$$
$$= \frac{25+10h+h^2-3}{5+h} - \frac{22}{5}$$
$$= \frac{22+10h+h^2}{5+h} - \frac{22}{5}$$
$$= \frac{5(22+10h+h^2) - 22(5+h)}{5(5+h)}$$
$$= \frac{28h+5h^2}{5(5+h)}$$
$$= h\cdot\frac{(28+5h)}{5(5+h)}$$

So

$$\frac{1}{h}\Delta_5 f(h) = \frac{1}{h}\cdot h\frac{(28+5h)}{5(5+h)} = \frac{28+5h}{5(5+h)}$$

c) $f(x) = 3\cos(x)$, $x_0 = \frac{\pi}{4}$

$$\Delta_{\frac{\pi}{4}} f(h) = f(\tfrac{\pi}{4}+h) - f(\tfrac{\pi}{4})$$
$$= 3\cos(\tfrac{\pi}{4}+h) - 3\cos(\tfrac{\pi}{4})$$
$$= 3\cos(\tfrac{\pi}{4})\cos(h) - 3\sin(\tfrac{\pi}{4})\sin(h) - 3\cos(\tfrac{\pi}{4})$$
$$= \frac{3\sqrt{2}}{2}\cos(h) - \frac{3\sqrt{2}}{2}\sin(h) - \frac{3\sqrt{2}}{2}$$
$$= -\frac{3\sqrt{2}}{2}(1-\cos(h)) - \frac{3\sqrt{2}}{2}\sin(h)$$

So

$$\frac{1}{h}\Delta_{\frac{\pi}{4}} f(h) = -\frac{3\sqrt{2}}{2}\frac{(1-\cos(h))}{h} - \frac{3\sqrt{2}}{2}\frac{\sin(h)}{h}$$

d) $f(x) = \sqrt{x}$, $x_0 = 9$

$$\Delta_9 f(h) = f(9+h) - f(9)$$
$$= \sqrt{9+h} - \sqrt{9}$$
$$= \sqrt{9+h} - 3$$
$$= (\sqrt{9+h} - 3)\cdot\frac{\sqrt{9+h}+3}{\sqrt{9+h}+3}$$
$$= \frac{9+h-9}{\sqrt{9+h}+3}$$
$$= \frac{h}{\sqrt{9+h}+3}$$

So

$$\frac{1}{h}\Delta_9 f(h) = \frac{1}{h}\cdot\frac{h}{\sqrt{9+h}+3}$$
$$= \frac{1}{\sqrt{9+h}+3}.$$

e) $f(x) = \exp(x)$, $x_0 = 1$

$$\Delta_1 f(h) = f(1+h) - f(1)$$
$$= \exp(1+h) - \exp(1)$$
$$= e^{1+h} - e$$
$$= e(e^h - 1)$$

So

$$\frac{1}{h}\Delta_1 f(h) = e\frac{(e^h - 1)}{h}$$

f) $f(x) = 4$, $x_0 = 1$

$$\Delta_1 f(h) = f(1+h) - f(1)$$
$$= 4 - 4$$
$$= 0.$$

So

$$\frac{1}{h}\Delta_1 f(h) = \frac{1}{h}\cdot 0$$
$$= 0.$$

2. For each function f and x_0 given below, set up a limit to determine $f'(x_0)$.

a) $f(x) = -2x^2 + x + 3$, $x_0 = 2$

$$\lim_{h\to 0} \frac{1}{h}\Delta_2 f(h) = \lim_{h\to 0}(-7-2h)$$
$$= -7,$$

so $f'(2) = -7.$

b) $f(x) = \frac{x^2-3}{x}$, $x_0 = 5$

$$\lim_{h\to 0} \frac{1}{h}\Delta_5 f(h) = \lim_{h\to 0} \frac{28+5h}{5(5+h)}$$
$$= \frac{28}{25},$$

So $f'(5) = \frac{28}{25}.$

c) $f(x) = 3\cos(x)$, $x_0 = \frac{\pi}{4}$

$$\lim_{h\to 0} \frac{1}{h}\Delta_{\frac{\pi}{4}} f(h) = \lim_{h\to 0} \frac{-3\sqrt{2}}{2}\frac{(1-\cos(h))}{h} - \frac{3\sqrt{2}}{2}\frac{\sin(h)}{h}$$

$$= 0 - \frac{3\sqrt{2}}{2}\cdot 1$$

$$= -\frac{3\sqrt{2}}{2},$$

So $f'(\frac{\pi}{4}) = -\frac{3\sqrt{2}}{2}$.

d) $f(x) = \sqrt{x}$, $x_0 = 9$

$$\lim_{h\to 0}\frac{1}{h}\Delta_9 f(h) = \lim_{h\to 0}\frac{1}{\sqrt{9+h}+3}$$

$$= \frac{1}{\sqrt{9}+3}$$

$$= \frac{1}{6},$$

So $f'(9) = \frac{1}{6}$.

e) $f(x) = \exp(x)$, $x_0 = 1$

$$\lim_{h\to 0}\frac{1}{h}\Delta_1 f(h) = \lim_{h\to 0} e\,\frac{(e^h - 1)}{h}$$

$$= e\cdot 1$$

since $\lim_{*\to 0}\frac{e^* - 1}{*}$

$$= e,$$

So $f'(1) = e$.

f) $f(x) = 4$, $x_0 = 1$

$$\lim_{h\to 0}\frac{1}{h}\Delta_1 f(h) = \lim_{h\to 0} 0$$

$$= 0,$$

So $f'(1) = 0$.

3. For each function f that is given below, use limits to calculate $f'(x)$ for every x where this limit exists.

a) $f(x) = 4$

$$\Delta_x f(h) = f(x+h) - f(x)$$
$$= 4 - 4$$
$$= 0$$
$$\lim_{h\to 0} \frac{1}{h} \Delta_x f(h) = \lim_{h\to 0} 0 = 0,$$
so $f'(x) = 0$.

b) $f(x) = 5x$

$$\Delta_x f(h) = f(x+h) - f(x)$$
$$= 5(x+h) - 5x$$
$$= 5x + 5h - 5x$$
$$= 5h,$$
so $f'(x) = \lim_{h\to 0} \frac{1}{h} \Delta_x f(h) = \lim_{h\to 0} \frac{1}{h} \cdot 5h = 5.$

c) $f(x) = -2x^2 + x + 3$

$$\Delta_x f(h) = -2(x+h)^2 + (x+h) + 3 + 2x^2 - x - 3$$
$$= -2x^2 - 4xh - 2h^2 + x + h + 3 + 2x^2 - x - 3$$
$$= -4xh + h - 2h^2$$
$$= h(-4x + 1 - 2h),$$
$$f'(x) = \lim_{h\to 0} \frac{1}{h} h(-4x + 1 - 2h)$$
$$= \lim_{h\to 0} (-4x + 1 - 2h)$$
$$= -4x + 1.$$

d) $f(x) = \frac{x^2 - 3}{x}$

$$\Delta_x f(h) = \frac{(x+h)^2 - 3}{x+h} - \frac{x^2 - 3}{x}$$
$$= \frac{x^2 + 2xh + h^2 - 3}{x+h} - \frac{x^2 - 3}{x}$$
$$= \frac{x(x^2 + 2xh + h^2 - 3) - (x+h)(x^2 - 3)}{x(x+h)}$$
$$= \frac{x^3 + 2x^2h + xh^2 - 3x - x^3 + 3x - x^2h + 3h}{x(x+h)}$$
$$= \frac{x^2h + xh^2 + 3h}{x(x+h)}$$
$$= h\frac{(x^2 + xh + 3)}{x(x+h)}$$
$$f'(x) = \lim_{h\to 0} \frac{1}{h} \cdot h\frac{(x^2 + xh + 3)}{x(x+h)} = \lim_{h\to 0} \frac{x^2 + xh + 3}{x(x+h)} \underset{x\neq 0}{=} \frac{x^2 + 3}{x^2}$$

e) $f(x) = \sqrt{x}$

$$\Delta_x f(h) = \sqrt{x+h} - \sqrt{x}$$
$$= (\sqrt{x+h} - \sqrt{x}) \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$
$$= \frac{x + h - x}{\sqrt{x+h} + \sqrt{x}}$$
$$= \frac{h}{\sqrt{x+h} + \sqrt{x}}$$

$$f'(x) = \lim_{h\to 0} \frac{1}{h} \cdot \frac{h}{\sqrt{x+h} + \sqrt{x}}$$
$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$
$$= \frac{1}{\sqrt{x} + \sqrt{x}}$$
$$= \frac{1}{2\sqrt{x}}.$$

4. Take f to be the function that is given by

$$f(x) = \cos(5x^2 + 1).$$

Show that f is differentiable for each x in $\mathbb{R}$ and determine $f'(x)$.

$$\Delta_x f(h) = f(x+h) - f(x)$$
$$= \cos(5(x+h)^2 + 1) - \cos(5x^2+1)$$
$$= \cos(5x^2+1+10xh+5h^2) - \cos(5x^2+1)$$
$$= \cos(5x^2+1)\cos(10xh+5h^2) - \sin(5x^2+1)\sin(10xh+5h^2) - \cos(5x^2+1)$$
$$= -\cos(5x^2+1)(1-\cos(10xh+5h^2)) - \sin(5x^2+1)\sin(10xh+5h^2),$$

$$\frac{1}{h}\Delta_x f(h) = -\cos(5x^2+1)\left(\frac{1-\cos(10xh+5h^2)}{h}\right) - \sin(5x^2+1)\frac{\sin(10xh+5h^2)}{h},$$
$$= -\cos(5x^2+1)\left(\frac{1-\cos(h(10x+5h))}{h}\right) - \sin(5x^2+1)\frac{\sin(h(10x+5h))}{h}$$
$$= -\cos(5x^2+1)(10x+5h)\left(\frac{1-\cos(h(10x+5h))}{h(10x+5h)}\right) - \sin(5x^2+1)(10x+5h)\frac{\sin(h(10x+5h))}{h(10x+5h)}$$

so

$$f'(x) = \lim_{h\to 0}\frac{1}{h}\Delta_x f(h)$$
$$= -\cos(5x^2+1)\lim_{h\to 0}(10x+5h)\lim_{h\to 0}\frac{1-\cos(h(10x+5h))}{h(10x+5h)} - \sin(5x^2+1)\lim_{h\to 0}(10x+5h)\lim_{h\to 0}\frac{\sin(h(10x+5h))}{h(10x+5h)}$$
$$= -\cos(5x^2+1)\cdot 10x\cdot 0 - \sin(5x^2+1)\cdot 10x\cdot 1$$
$$= 10x\sin(5x^2+1).$$

5. Take f to be the function that is given by

$$f(x) = \mathrm{e}^{x^2+4x}.$$

Show that f is differentiable for each x in $\mathbb{R}$ and determine $f'(x)$.

$$\begin{aligned}
\Delta_x f(h) &= f(x+h) - f(x) \\
&= \exp((x+h)^2 + 4(x+h)) - \exp(x^2+4x) \\
&= \exp(x^2+4x+2xh+4h+h^2) - \exp(x^2+4x) \\
&= \exp(x^2+4x)\exp(2xh+4h+h^2) - \exp(x^2+4x) \\
&= \exp(x^2+4x)\left(\exp(h(2x+4+h)) - 1\right),
\end{aligned}$$

$\exp(A+B) = \exp(A)\exp(B)$

so

$$\begin{aligned}
\lim_{h\to 0} \frac{1}{h}\Delta_x f(h) &= \lim_{h\to 0} \exp(x^2+4x)\left(\frac{\exp(h(2x+4+h))-1}{h}\right) \\
&= \lim_{h\to 0} \exp(x^2+4x)(2x+4+h)\left(\frac{\exp(h(2x+4+h))-1}{h(2x+4+h)}\right) \\
&= \exp(x^2+4x)(2x+4)\cdot 1 \\
&= (2x+4)e^{x^2+4x}.
\end{aligned}$$

$\lim_{h\to 0} \frac{\exp(h)-1}{h} = 1$

Thus, $f'(x) = (2x+4)e^{x^2+4x}$.

6. Take c to be the path that is given for each t by

$$c(t) = (t^3 - 5t, 6\sin(5t)).$$

Use limits to directly calculate $c'(t)$.

$$c(t+h) - c(t) = \left((t+h)^3 - 5(t+h), 6\sin(5(t+h))\right) - (t^3 - 5t, 6\sin(5t))$$

$$= (t^3 + 3t^2h + 3th^2 + h^3 - 5t - 5h, 6\sin(5t+5h)) - (t^3 - 5t, 6\sin(5t))$$ combine

$$= \langle 3t^2h + 3th^2 + h^3 - 5h, 6\sin(5t+5h) - 6\sin(5t)\rangle$$

$$= \langle 3t^2h + 3th^2 + h^3 - 5h, 6\sin(5t)\cos(5h) + 6\cos(5t)\sin(5h) - 6\sin(5t)\rangle$$

$$= \langle h(3t^2 + 3th + h^2 - 5), -6\sin(5t)(1 - \cos(5h)) + 6\cos(5t)\sin(5h)\rangle$$

Hence

$$\lim_{h\to 0} \frac{c(t+h) - c(t)}{h} = \lim_{h\to 0} \left\langle \frac{1}{h}\cdot h(3t^2 + 3th + h^2 - 5), -6\sin(5t)\left(\frac{1-\cos(5h)}{h}\right) + 6\cos(5t)\frac{\sin(5h)}{h}\right\rangle$$

$$= \lim_{h\to 0} \left\langle 3t^2 + 3th + h^2 - 5, -30\sin(5t)\left(\frac{1-\cos(5h)}{5h}\right) + 30\cos(5t)\frac{\sin(5h)}{5h}\right\rangle$$

$$= \left\langle 3t^2 + 0 + 0 - 5, -30\sin(5t)\cdot 0 + 30\cos(5t)\cdot 1\right\rangle$$

$$= \langle 3t^2 - 5, 30\cos(5t)\rangle.$$

Thus, $c'(t) = \langle 3t^2 - 5, 30\cos(5t)\rangle$.