1. For each function f that is given below, classify all zeros of f by finding the largest natural number n and a real number x_0 so that f is $O((x-x_0)^n)$.

a) $f(x) = (x-2)(x+6)^2(x-10)^4$

f is $O(x-2)$

$f(x) = \underbrace{(x+6)^2(x-10)^4}_{\eta(x)} \cdot (x-2)$

η is bounded near $x_0 = 2$

f is $O(x+6)^2$

$f(x) = \underbrace{(x-2)(x-10)^4}_{\eta(x)} \cdot (x+6)^2$

η is bounded near $x_0 = -6$

f is $O(x-10)^4$

$f(x) = \underbrace{(x-2)(x+6)^2}_{\eta(x)} \cdot (x-10)^4$

η is bounded near $x_0 = 10$

b) $f(x) = \frac{(x-4)^6(x+5)^3}{x+7}$

f is $O(x-4)^6$

$f(x) = \underbrace{\frac{(x+5)^3}{x+7}}_{\eta(x)} \cdot (x-4)^6$

η is bounded near $x_0 = 4$

f is $O(x+5)^3$

$f(x) = \underbrace{\frac{(x-4)^6}{x+7}}_{\eta(x)} \cdot (x+5)^3$

η is bounded near $x_0 = -5$.

2. For each function f and x_0 that is given below, find the largest natural numbers m and n so that f is $O((x-x_0)^m)$ and $o((x-x_0)^n)$.

a) $f(x) = (x-2)(x+6)^2(x-10)^4$ and $x_0 = 2$

f is $O(x-2)$

$\underbrace{(x+6)^2(x-10)^4}_{\text{bounded near } x_0=2} \cdot (x-2)$

f is $O(x+6)^2$, $o(x+6)$

$\underbrace{(x-2)(x-10)^4}_{\text{bounded near } x_0=-6} \cdot (x+6)^2$ $\quad$ $\underbrace{(x-2)(x-10)^4(x+6)}_{\text{goes to } 0 \text{ as } x\to -6} \cdot (x+6)$

f is $O(x-10)^4$, $o(x-10)^3$

$\underbrace{(x-2)(x+6)^2}_{\text{bounded near } x_0=10} \cdot (x-10)^4$ $\quad$ $\underbrace{(x-2)(x+6)^2(x-10)}_{\text{goes to } 0 \text{ as } x\to 10} \cdot (x-10)^3$

b) $f(x) = (x+5)\sin((x+6)^2)$ and $x_0 = -6$

f is $O(x+5)$

$\underbrace{\sin((x+6)^2)}_{\text{bounded near } x_0=-5} \cdot (x+5)$

f is $O(x+6)^2$

$\underbrace{(x+5)\frac{\sin((x+6)^2)}{(x+6)^2}}_{\text{bounded near } x_0=-6 \text{ (excluding } -6)} \cdot (x+6)^2$

since

$$\lim_{x_0\to -6} \frac{\sin((x+6)^2)}{(x+6)^2} = 1$$

$o(x+6)$

$\underbrace{(x+5)\frac{\sin((x+6)^2)}{(x+6)^2}(x+6)}_{\text{goes to } 0 \text{ as } x\to -6} \cdot (x+6)$

c) $f(x) = (x+6)\sin(x+6)$ and $x_0 = -6$

f is $O(x+6)^2$

$\underbrace{\frac{\sin(x+6)}{x+6}}_{\text{bounded near } x_0=-6 \text{ (exclude } -6)} \cdot (x+6)^2$

since

$$\lim_{x_0\to -6} \frac{\sin(x+6)}{x+6} = 1$$

f is $o(x+6)$

$\underbrace{\sin(x+6)}_{\text{goes to } 0 \text{ as } x \text{ goes to } -6} \cdot (x+6)$

3. For each function f and each real number x_0, write a formula for $\Delta_{x_0} f(h)$ and simplify the expression as much as possible.

a) $f(x) = -x^2 + 2x + 1$ and $x_0 = -2$

$$\Delta_{-2} f(h) = f(-2+h) - f(-2)$$
$$= -(-2+h)^2 + 2(-2+h) + 1 + 7 \quad \text{expand}$$
$$= -4 + 4h - h^2 - 4 + 2h + 8 \quad \text{expand}$$
$$= 6h - h^2 \quad \text{simplify}$$
$$= h(6-h). \quad \text{factor } h$$

b) $f(x) = \cos(x)$ and $x_0 = \frac{\pi}{4}$

$$\Delta_{\frac{\pi}{4}} f(h) = f\left(\tfrac{\pi}{4} + h\right) - f\left(\tfrac{\pi}{4}\right)$$
$$= \cos\left(\tfrac{\pi}{4} + h\right) - \cos\left(\tfrac{\pi}{4}\right) \quad \text{sum of angles}$$
$$= \cos\left(\tfrac{\pi}{4}\right)\cos(h) - \sin\left(\tfrac{\pi}{4}\right)\sin(h) - \cos\left(\tfrac{\pi}{4}\right)$$
$$= \tfrac{\sqrt{2}}{2}\cos(h) - \tfrac{\sqrt{2}}{2}\sin(h) - \tfrac{\sqrt{2}}{2} \quad \text{evaluate}$$
$$= -\tfrac{\sqrt{2}}{2}(1 - \cos(h)) - \tfrac{\sqrt{2}}{2}\sin(h) \quad \text{regroup } h$$

c) $f(x) = \sqrt{x-1}$ and $x_0 = 5$

$$\Delta_5 f(h) = f(5+h) - f(5)$$
$$= \sqrt{5+h-1} - \sqrt{5-1}$$
$$= \sqrt{h+4} - 2 \quad \text{simplify}$$
$$= (\sqrt{h+4} - 2) \cdot \frac{(\sqrt{h+4} + 2)}{(\sqrt{h+4} + 2)} \quad \text{multiply and divide by } \sqrt{h+4} + 2$$
$$= \frac{h + 4 - 4}{(\sqrt{h+4} + 2)} \quad \text{expand numerator}$$
$$= \frac{h}{(\sqrt{h+2} + 2)}$$

4. Take f and g to be the functions that are given by

$$g(x) = 4 + \sin(x) \quad \text{and} \quad f(x) = 1 - \cos(x - 4).$$

Perform the appropriate calculations in order to justify the following statements.

a) $g - g(0)$ is $O(x)$

$g(0) = 4 + \sin(0) = 4,$

$$\begin{aligned}(g - g(0))(x) &= g(x) - g(0) \\ &= 4 + \sin(x) - 4 \\ &= \sin(x) \\ &= \underbrace{\frac{\sin(x)}{x}}\cdot x\end{aligned}$$

bounded near 0 (exclude 0) since $\lim\limits_{x\to 0} \frac{\sin(x)}{x} = 1.$

b) f is $o(x - g(0))$

f is $o(x-4)$ is equivalent to showing that $\lim\limits_{x\to 4} \frac{f(x)}{x-4} = 0.$

$$\lim_{x\to 4} \frac{f(x)}{x-4} = \lim_{x\to 4} \frac{1-\cos(x-4)}{x-4}$$

since $\lim\limits_{x\to 0} \frac{1-\cos(x)}{x} = 0$

$$= 0.$$

c) $f \circ g$ is $o(x)$

$f \circ g$ is $o(x)$ is equivalent to showing that $\lim\limits_{x\to 0} \frac{(f\circ g)(x)}{x} = 0.$

$$\begin{aligned}\lim_{x\to 0} \frac{(f\circ g)(x)}{x} &= \lim_{x\to 0} \frac{f(g(x))}{x} \\ &= \lim_{x\to 0} \frac{1-\cos(g(x)-4)}{x} \\ &= \lim_{x\to 0} \frac{1-\cos(4+\sin(x)-4)}{x} \\ &= \lim_{x\to 0} \frac{1-\cos(\sin(x))}{x} \\ &= \lim_{x\to 0} \left(\frac{1-\cos(\sin(x))}{\sin(x)}\right)\cdot\left(\frac{\sin(x)}{x}\right) \\ &= 0 \cdot 1 \\ &= 0.\end{aligned}$$

multiply and divide by $\sin(x)$ and regroup h