1. Take f to be a function with the property that

$$\lim_{x \to -1^-} f(x) = 5 = \lim_{x \to -1^+} f(x).$$

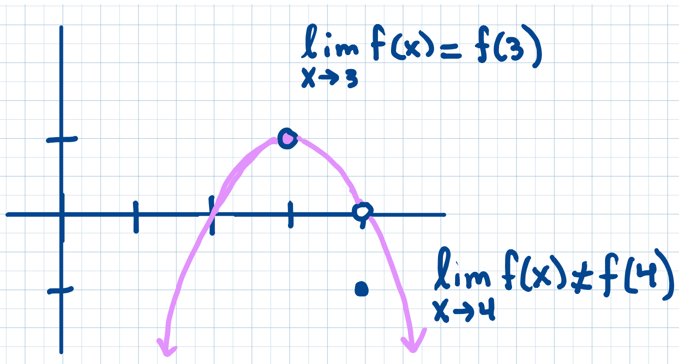
Determine the value of $f(-1)$ so that f is continuous at -1.

Continuous at -1 if and only if $\lim_{x \to -1} f(x) = f(-1)$.

Since left and right sided limit exist, $\lim_{x \to -1} f(x)$ exists and equals 5.

So $f(-1)$ must equal 5.

2. Sketch a function f that is continuous at $x = 3$ but not continuous at $x = 4$.



$$f(x) = \begin{cases} -(x-3)^2 + 1 & \text{if } x < 4 \\ -1 & \text{if } x = 4 \\ -(x-3)^2 + 1 & \text{if } x > 4 \end{cases}$$

3. Show that the function f is continuous at 4, where f is given by

$$f(x) = \begin{cases} \dfrac{1 - \cos(x-4)}{x-4} & \text{if } x \neq 4 \\ 2x - 8 & \text{if } x = 4. \end{cases}$$

Show limit at $x=4$ exists:

$\lim_{x \to 4} f(x) = \lim_{x \to 4} \dfrac{1 - \cos(x-4)}{x-4} = 0$ since $\lim_{* \to 0} \dfrac{1 - \cos(*)}{*} = 0.$

Determine $f(4)$:

$f(4) = 2(4) - 8 = 8 - 8 = 0.$

Because $\lim_{x \to 4} f(x) = f(4)$, f is continuous.

4. Show that the function f is continuous at 1, where f is given by

$$f(x) = \begin{cases} x^2+5 & \text{if } x<1 \\ 6 & \text{if } x=1 \\ \log_2(x+63) & \text{if } x>1. \end{cases}$$

Show $\lim_{x\to 1} f(x)$ exist by showing $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$.

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(x^2+5) = 1+5=6$ and $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \log_2(x+63) = \log_2(64) = 6$

So $\lim_{x\to 1} f(x) = 6$, which equals $f(1)$. Thus, f is continuous at $x=1$.

5. Take f to be the function that is given by

$$f(x) = \frac{1-\cos(x-4)}{x-4}.$$

Determine the maximal domain of f and find a continuous extension of f to all of $\mathbb{R}$.

Maximal domain is $D(f) = (-\infty, 4) \cup (4, \infty)$.

Note that $\lim_{x\to 4} f(x) = 0$, so a continuous extension of f to all of $\mathbb{R}$ is

$$\tilde{f}(x) = \begin{cases} \dfrac{1-\cos(x-4)}{x-4} & \text{if } x \neq 4 \\ 0 & \text{if } x=4. \end{cases}$$

6. Carefully show that the function f is not continuous at 2, where f is given by

$$f(x) = \begin{cases} \exp_5(x) & \text{if } x \le 2 \\ 2x+5 & \text{if } x>2 \end{cases} \quad \text{where} \quad \exp_5(x) = 5^x.$$

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \exp_5(x) = \exp_5(2) = 5^2 = 25$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (2x+5) = 2(2)+5 = 9.$

Since $\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$, the limit does not exist at 2, so f is not continuous at 2.

7. Take f and g to be functions that are defined on

$$\mathcal{D}(f) = (-\infty, \infty) \quad \text{and} \quad \mathcal{D}(g) = [-6, \infty)$$

and that are continuous on

$$S_f = (-\infty, 3) \cup (3, 6) \cup (6, \infty) \quad \text{and} \quad S_g = [-6, -2) \cup (-2, 0) \cup (0, 2) \cup (2, \infty),$$

respectively.

a. Determine whether f is defined at $x = 3$.

Yes, since 3 is in $D(f)$.

b. Determine whether f is continuous at $x = 3$ and $x = 9$.

Not continuous at 3 since it is not in S_f.

Continuous at 9 since it is in S_f.

c. Determine whether g is defined at $x = -7$.

No, since -7 is not in $D(g)$.

d. Determine whether g continuous at $x = -2$ and $x = 2$.

No, not continuous at -2 and 2 because neither is in S_g.

e. Determine the maximal set on which $f + g$ is continuous on.

Maximal set on which $f+g$ is continuous on is

$$S_f \cap S_g = [-6,-2) \cup (-2,0) \cup (0,2) \cup (2,3) \cup (3,6) \cup (6,\infty).$$

f. Determine the maximal set on which fg is continuous on.

Maximal set on which fg is continuous on is

$$S_f \cap S_g = [-6,-2) \cup (-2,0) \cup (0,2) \cup (2,3) \cup (3,6) \cup (6,\infty).$$

8. Decompose the function f into sums, products, and quotients of continuous functions to find a set S on which f is continuous, where f is given by

$$f(x) = \frac{\sqrt{10-x}}{x^2-6} + \ln(x) + \sin(x).$$

Take $a(x) = \sqrt{10-x}$, $b(x) = x^2 - 6$, $c(x) = \ln(x)$ and $d(x) = \sin(x)$ so that

$$f = \frac{a}{b} + c + d.$$

a is continuous on $S_a = (-\infty, 10]$ and b is continuous on $S_b = (-\infty, \infty)$. b is zero on $Z(b) = \{-\sqrt{6}, \sqrt{6}\}$. Thus $\frac{a}{b}$ is continuous on

$$(S_a \cap S_b) \setminus Z(b) = (-\infty, -\sqrt{6}) \cup (-\sqrt{6}, \sqrt{6}) \cup (\sqrt{6}, 10].$$

c is continuous on $S_c = (0, \infty)$ and d is continuous on $S_d = (-\infty, \infty)$. Thus $c + d$ is continuous on

$$S_c \cap S_d = (0, \infty).$$

f is continuous on

$$((S_a \cap S_b) Z(b)) \cap (S_c \cap S_d) = (0, \sqrt{6}) \cup (\sqrt{6}, 10].$$

9. Write the function f as a composite function to determine $\lim_{x\to 4} f(x)$, where f is given by

$$f(x) = \cos\left(\frac{1-\cos(x-4)}{x-4}\right).$$

Rewrite f as $f = \cos \circ g$ where $g(x) = \dfrac{1-\cos(x-4)}{x-4}$.

Determine $\lim_{x\to 4} g(x)$:

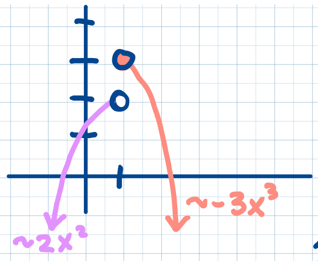
$$\lim_{x\to 4} g(x) = \lim_{x\to 4}\left(\frac{1-\cos(x-4)}{x-4}\right) = 0.$$

Because cos is continuous on $\mathbb{R}$, it is continuous at 0. So, by the limit law for composite functions, we have that

$$\lim_{x\to 4} f(x) = \cos\left(\lim_{x\to 4} \frac{1-\cos(x-4)}{x-4}\right) = \cos(0) = 1.$$

10. Construct a function that is continuous everywhere except at 1, that is strictly increasing to the left of 1, is asymptotically equal to $-2x^2$ to the left, is strictly decreasing to the right of 1, is right continuous at 1, is asymptotically equal to $-3x^3$, and has the property that

$$\lim_{x\to 1^-} f(x) = 2 \quad \text{and} \quad \lim_{x\to 1^+} f(x) = 3.$$



$$f(x) = \begin{cases} -2(x-1)^2+2 & \text{if } x<1 \\ -3(x-1)^3+3 & \text{if } x \geq 1 \end{cases}$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (-2(x-1)^2+2) = -2(1-1)^2+2 = 2$$

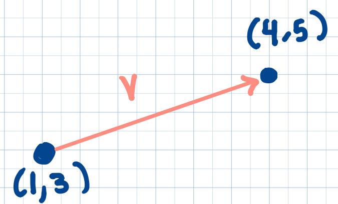
$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (-3(x-1)^3+3) = -3(1-1)^3+3 = 3 = f(1)$$ so f is right continuous.

$$\lim_{x\to -\infty} \frac{f(x)}{-2x^2} = \lim_{x\to -\infty} \frac{-2(x-1)^2+2}{-2x^2} = \lim_{x\to -\infty} \frac{-2x^2+4x}{-2x^2} = 1$$ so $f \sim -2x^2$ to the left.

$$\lim_{x\to \infty} \frac{f(x)}{-3x^3} = \lim_{x\to \infty} \frac{-3(x-1)^3+3}{-3x^3} = \lim_{x\to\infty} \frac{-3x^3+9x-9x^2+6}{-3x^3} = 1$$ so $f \sim -3x^3$ to the right.

11. Construct a continuous path c with domain $[0,\infty)$ that describes the position of a particle that moves to the right on the line segment from $(1,3)$ to $(4,5)$, is at $(1,3)$ at time 0, is never at the same point at different time points, that never reaches $(4,5)$, but that has the property that

$$\lim_{t\to\infty} \|(4,5) - c(t)\| = 0.$$



$$V = (4,5) - (1,3) = \langle 3,2 \rangle$$

$$c(t) = d(t)\langle 3,2\rangle + (1,3)$$

Where d satisfies

- $d(0) = 0$
- $\lim_{t\to\infty} d(t) = 1.$

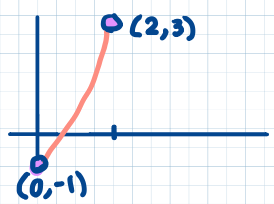
So,

$$c(t) = \frac{t}{t+1}\langle 3,2\rangle + (1,3).$$

12. Take f to be the polynomial that is given by

$$f(x) = x^5 - 4x^3 + 2x - 1.$$

Show that f has at least one real root by using the intermediate value theorem.



Take $I = [0,2]$. The function f is continuous on I since f is a polynomial and it is continuous on $\mathbb{R}$. Also

$$f(0) = 0^5 - 4(0)^3 + 2(0) - 1 = -1,$$
$$f(2) = 2^5 - 4(2)^3 + 2(2) - 1 = 3.$$

Because 0 is in $[f(0), f(2)] = [-1, 3]$, by the intermediate value theorem there is a z in $(0,2)$ so that $f(z) = 0$. This shows that f has at least one root.

13. Use the bisection method to approximate a solution to the equation

$$x^2 = 7$$

to within an error of no greater than $\frac{1}{10}$.

Take $f(x) = x^2 - 7$ and $I = [2,3]$. Since $f(2) < 0$ and $f(3) > 0$, by the intermediate Value theorem, $f(x) = 0$ has a solution in $[2,3]$:

Take $c = \frac{2+3}{2} = \frac{5}{2}$. Since $f(\frac{5}{2}) = -\frac{3}{2} < 0$, update interval to $[\frac{5}{2}, 3]$.

Take $c = \frac{\frac{5}{2}+3}{2} = \frac{11}{4}$. Since $f(\frac{11}{4}) = \frac{9}{16} > 0$, update interval to $[\frac{5}{2}, \frac{11}{4}]$.

Take $c = \frac{\frac{5}{2}+\frac{11}{4}}{2} = \frac{21}{8}$. Since $f(\frac{21}{8}) = -\frac{7}{64} < 0$, update interval to $[\frac{21}{8}, \frac{11}{4}]$.

Take $c = \frac{\frac{21}{8}+\frac{11}{4}}{2} = \frac{43}{16}$. Since $f(\frac{43}{16}) = \frac{57}{256} > 0$, update interval to $[\frac{21}{8}, \frac{43}{16}]$.

Take $c = \frac{\frac{21}{8}+\frac{43}{16}}{2} = \frac{85}{32}$. Since $f(\frac{85}{32}) = \frac{57}{1024} > 0$, update interval to $[\frac{21}{8}, \frac{85}{32}]$.

Take $c = \frac{\frac{21}{8}+\frac{85}{32}}{2} = \frac{169}{64}$. Since $f(\frac{169}{64}) = -\frac{111}{4096} < 0$ update interval to $[\frac{169}{64}, \frac{85}{32}]$.

Stop because

$$\left|f\left(\tfrac{169}{64}\right)\right| = \frac{111}{4096} < \frac{1}{10} \quad \text{and} \quad \left|f\left(\tfrac{85}{32}\right)\right| = \frac{57}{1024} < \frac{1}{10}.$$

So,

$$\frac{111}{4096} \leq \sqrt{7} \leq \frac{57}{1024}.$$