1. Take f to be the function given by

$$f(x) = \begin{cases} \dfrac{x^2-3x-10}{x-5} & \text{if } x \neq 5 \\ 12 & \text{if } x = 5. \end{cases}$$

Determine $\lim_{x\to 5} f(x)$ and whether the limit depends on $f(5)$.

For any sequence (x_n) that converges to 5 and $x_n \neq 5$ for all n, we have

$$f(x_n) = \frac{(x_n)^2 - 3x_n - 10}{x_n - 5} = \frac{(x_n-5)(x_n+2)}{x_n-5} = x_n + 2$$

So, $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty}(x_n+2) = 5+2 = 7.$

Hence $\lim_{x\to 5} f(x) = 7$. The limit did not depend on $f(5)$.

2. Take f to be a function with the property that $\lim_{x\to 2} f(x) = 7$. Determine $\lim_{x\to 2^-} f(x)$ and $\lim_{x\to 2^+} f(x)$.

Because $\lim_{x\to 2} f(x) = 7$, the left-hand and right hand limit at 2 exists and must equal the limit at 2. Thus

$$\lim_{x\to 2^-} f(x) = 7 \quad \text{and} \quad \lim_{x\to 2^+} f(x) = 7.$$

3. Show that the following limit does not exist:

$$\lim_{x\to 6} \frac{|x-6|}{x-6}.$$

Notice that

$$\frac{|x-6|}{x-6} = \begin{cases} \dfrac{x-6}{x-6} & \text{if } x > 6 \\ \dfrac{-(x-6)}{x-6} & \text{if } x < 6. \end{cases}$$

So $\lim_{x\to 6^+} \frac{|x-6|}{x-6} = \lim_{x\to 6^+} \frac{x-6}{x-6} = 1$ and $\lim_{x\to 6^-} \frac{|x-6|}{x-6} = \lim_{x\to 6^-} \frac{-(x-6)}{x-6} = -1.$

$x\to 6^+ \Rightarrow x>6$ $\qquad$ $x\to 6^- \Rightarrow x<6$

The limit at 6 does not exist because the left-hand and right-hand limits are not equal.

4. Create a function for which $\lim_{x\to 3^+} f(x)$ and $\lim_{x\to 3^-} f(x)$ exist, but $\lim_{x\to 3} f(x)$ does not exist.

Here is one example:

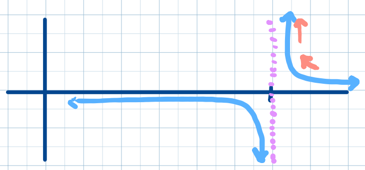
$$f(x) = \begin{cases} x-3 & \text{if } x \le 3 \\ 4 & \text{if } x > 3. \end{cases}$$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} 4 = 4$ and $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (x-3) = 0$,

So both one-sided limits exists. However, the limit at 3 does not exist Since the one-sided limits are not equal.

5. Determine the way in which the following limits diverge:

a) $\lim_{x\to 6^+} \frac{1}{(x-6)^3}$



$x \to 6^+$ means $x-6>0$ and $x-6 \to 0$,

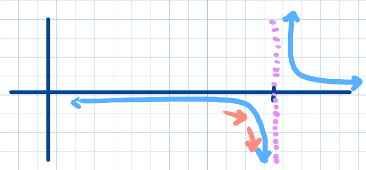
So $\frac{1}{(x-6)^3} \to \infty$.

Thus,

$$\lim_{x\to 6^+} \frac{1}{(x-6)^3} = \infty$$

diverges to ∞.

b) $\lim_{x\to 6^-} \frac{1}{(x-6)^3}$



$x \to 6^-$ means $x-6<0$ and $x-6 \to 0$,

So $\frac{1}{(x-6)^3} \to -\infty$

Thus,

$$\lim_{x\to 6^-} \frac{1}{(x-6)^3} = -\infty$$

diverges to $-\infty$.

6. Take a, b, c, and d to be functions that are defined in an open interval that contains 2 and

$$\lim_{x\to 2} a(x) = 6, \quad \lim_{x\to 2} b(x) = -1, \quad \lim_{x\to 2} c(x) = 4, \text{and} \quad \lim_{x\to 2} d(x) = 2.$$

Use the limit laws to determine the following limits:

$$\lim_{x\to 2}\left((a(x))^2 b(x) + \frac{(x-3)c(x)}{d(x)+3}\right).$$

$$\lim_{x\to 2}\left((a(x))^2 \cdot b(x) + \frac{(x-3)\cdot c(x)}{d(x)+3}\right) = \lim_{x\to 2} a(x) \cdot \lim_{x\to 2} a(x) \cdot \lim_{x\to 2} b(x) + \frac{\lim_{x\to 2}(x-3) \cdot \lim_{x\to 2} c(x)}{\lim_{x\to 2} d(x) + \lim_{x\to 2} 3}$$

quotient
product
sum
limit laws

$$= 6 \cdot 6 \cdot (-1) + \frac{(-1)\cdot 4}{2+3}$$

evaluate limits

$$= -36 - \frac{4}{5}$$

$$= -\frac{184}{5}.$$

7. Determine the following limit and carefully justify your reasoning:

$$\lim_{x\to 3} \frac{\sin(4(x-3))}{6(x-3)}.$$

$x \to 3$ means $(x-3) \to 0$. So $4(x-3) \to 0$ and $\displaystyle\lim_{x\to 3} \frac{\sin(4(x-3))}{4(x-3)} = 1.$

Thus,

$$\lim_{x\to 3} \frac{\sin(4(x-3))}{6(x-3)} = \lim_{x\to 3}\left(\frac{4}{6} \cdot \frac{\sin(4(x-3))}{4(x-3)}\right)$$

$$= \lim_{x\to 3} \frac{2}{3} \cdot \lim_{x\to 3} \frac{\sin(4(x-3))}{4(x-3)}$$

$$= \frac{2}{3} \cdot 1$$

$$= \frac{2}{3}.$$

8. Use the squeeze theorem to determine the following limit: $\lim_{x\to 5} \sqrt{|x-5|}\sin\left(\frac{1}{x-5}\right)$.

For any $x \neq 5$, $-1 \leq \sin\left(\frac{1}{x-5}\right) \leq 1$ and $\sqrt{|x-5|} > 0$ imply that

$$-\sqrt{|x-5|} \leq \sqrt{|x-5|}\sin\left(\frac{1}{x-5}\right) \leq \sqrt{|x-5|} \text{ for all } x \neq 5.$$

Take g and h to be the functions given by

$$g(x) = -\sqrt{|x-5|} \text{ and } h(x) = \sqrt{|x-5|}.$$

Because

$g(x) \leq \sqrt{|x-5|}\sin\left(\frac{1}{x-5}\right) \leq h(x)$ for $x \neq 5$, $\lim_{x\to 5} g(x) = 0$ and $\lim_{x\to 5} h(x) = 0$,

the squeeze theorem implies that $\lim_{x\to 5} \sqrt{|x-5|}\sin\left(\frac{1}{x-5}\right) = 0$.

9. Take f to be the function given by

$$f(x) = \begin{cases} \dfrac{x^2+x-2}{x-1} & \text{if } x < 1 \\ 3x-1 & \text{if } x \geq 1. \end{cases}$$

a) Determine $\lim_{x\to 1^-} f(x)$.

$x \to 1^-$ means $x < 1$ so

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{x^2+x-2}{x-1}$$

$$= \lim_{x\to 1^-} \frac{(x+2)(x-1)}{x-1}$$

$$= \lim_{x\to 1^-} (x+2)$$

$$= 3.$$

b) Determine $\lim_{x\to 1^+} f(x)$.

$x \to 1^+$ means $x > 1$ so

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (3x-1)$$

$$= 3\lim_{x\to 1^+} x - \lim_{x\to 1^+} 1$$

$$= 3\cdot 1 - 1$$

$$= 2.$$

c) Determine whether $\lim_{x\to 1} f(x)$ exists and determine the limit if it exists.

Limit does not exist because $\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$.

10. Take a to be a real number and f to be the function given by

$$f(x) = \begin{cases} x^2 - 1 & \text{if } x \leq 4 \\ a\log_2(x) + 5 & \text{if } x > 4. \end{cases}$$

a) Determine $\lim_{x\to 4^-} f(x)$.

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} (x^2 - 1)$$

$$= 4^2 - 1$$

$$= 15.$$

b) Determine $\lim_{x\to 4^+} f(x)$.

$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} (a\log_2(x) + 5)$$

$$= a\log_2(4) + 5$$

$$= 2a + 5.$$

c) Determine a value for a so that $\lim_{x\to 4} f(x)$ exist.

Limit exists if and only if $\lim_{x\to 4^-} f(x) = \lim_{x\to 4^+} f(x)$. So,

$$15 = 2a + 5$$

$$10 = 2a$$

$$5 = a.$$

Thus, a needs to be equal to 5.

11. Calculate the following:

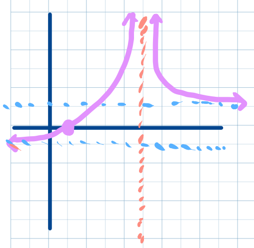
a) $\lim_{x\to 5} \frac{x-1}{|x-5|}$

$x \to 5$ means $|x-5| \to 0$ and $x-1 \to 4$.

Moreover, $|x-5| > 0$ for all $x \neq 5$, so we have that $\frac{1}{|x-5|} \to \infty$.

Since the limit of the numerator is positive, we have that

$$\lim_{x\to 5} \frac{x-1}{|x-5|} = \infty.$$



b) $\lim_{x\to\infty} \left(\sqrt{16x^2+2x} - 4x\right)$

$$\lim_{x\to\infty} \left(\sqrt{16x^2+2x} - 4x\right) = \lim_{x\to\infty} \left(\sqrt{16x^2+2x} - 4x\right) \cdot \frac{\left(\sqrt{16x^2+2x} + 4x\right)}{\left(\sqrt{16x^2+2x} + 4x\right)}$$

$$= \lim_{x\to\infty} \frac{16x^2+2x - 16x^2}{\sqrt{16x^2+2x} + 4x}$$ expand numerator

$$= \lim_{x\to\infty} \frac{2x}{\sqrt{16x^2+2x} + 4x}$$ simplify numerator

$$= \lim_{x\to\infty} \frac{2x}{\sqrt{16x^2+2x} + 4x} \cdot \frac{\frac{1}{x}}{\frac{1}{x}}$$ multiply and divide by $\frac{1}{x}$

$$= \lim_{x\to\infty} \frac{2}{\sqrt{16x^2+2x}\cdot\frac{1}{x} + 4}$$ distribute to numerator and denominator

$$= \lim_{x\to\infty} \frac{2}{\sqrt{16x^2+2x}\cdot\sqrt{\frac{1}{x^2}} + 4}$$ since $x > 0$, $\frac{1}{x} = \sqrt{\frac{1}{x^2}}$

$$= \lim_{x\to\infty} \frac{2}{\sqrt{16 + \frac{2}{x}} + 4}$$ distribute using $\sqrt{a}\cdot\sqrt{b} = \sqrt{a\cdot b}$

$$= \frac{2}{\sqrt{16+0} + 4}$$ evaluate limit $x\to\infty \Rightarrow \frac{2}{x} \to 0$

$$= \frac{2}{8}$$

$$= \frac{1}{4}.$$

12. Determine all horizontal and vertical asymptotes of the function f that is given by

$$f(x) = \begin{cases} 5^x + 1 & \text{if } x < -2 \\ \dfrac{5x+1}{(x+4)(x-4)} & \text{if } -2 \leq x < 4 \\ \arctan(-x) + \frac{5}{x+2} & \text{if } x \geq 4. \end{cases}$$

Horizontal asymptote are determined by taking limit as $x \to \pm\infty$

$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} (5^x+1) = 0+1 = 1,$ $\lim_{x\to-\infty} 5^x = 0$ HA of $y=0$.

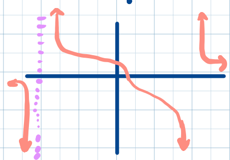
$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \left(\arctan(-x) + \frac{5}{x+2}\right) = -\frac{\pi}{2} + 0 = -\frac{\pi}{2},$ $\lim_{x\to\infty} \arctan(-x) = -\frac{\pi}{2}$ inputs < 0

So f has horizontal asymptotes at $y=1$ and $y=-\frac{\pi}{2}$.

Vertical asymptote are determined by identify a real number a where function diverges to $\pm\infty$ as $x \to a$, $x\to a^+$ or $x \to a^-$.

When $x<-2$, $f(x) = 5^x+1$. No VA here

When $-2 \leq x < 4$, $f(x) = \frac{5x+1}{(x+4)(x-4)}$. There is a pole at $x=4$, so $\lim_{x\to4^-} f(x) = -\infty$



When $x \geq 4$, $f(x) = \arctan(-x) + \frac{5}{x+2}$. No VA in the interval $[4,\infty)$.

So f has vertical asymptote at $x=4$.

13. Take f to be the rational function that is given by

$$f(x) = \frac{x^4(x+4)^6(x-1)^2(x-5)^2}{(1-4x)^3(x+3)^8}.$$

Find a monomial g so that f and g have the same asymptotic behavior at both ∞ and $-\infty$.

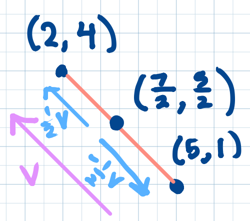
$$f \sim \frac{x^{14}}{-64x^{11}} = -\frac{1}{64}x^3.$$

Take g to be the monomial given by

$$g(x) = -\frac{1}{64}x^3.$$

Then f and g have the same asymptotic behavior.

14. Identify a path that describes the position in time of a particle that moves along the line segment L with endpoints $(2, 4)$ and $(5, 1)$, has domain equal to $\mathbb{R}$, is at the midpoint of L at time 0, moves only to the left, and reaches all points of L except the endpoints of L.



Take $V = (2,4) - (5,1) = \langle -3, 3\rangle$ and

$$c(t) = d(t)\tfrac{1}{2}V + \left(\tfrac{7}{2}, \tfrac{5}{2}\right)$$

where

$$d(0) = 0$$

$$\lim_{t\to\infty} d(t) = 1$$

$$\lim_{t\to-\infty} d(t) = -1$$

Here is an example:

$$d(t) = \frac{2}{\pi}\arctan(t).$$

Thus

$$c(t) = \frac{1}{\pi}\arctan(t)\langle -3, 3\rangle + \left(\frac{7}{2}, \frac{5}{2}\right).$$