1. Suppose that (θ_n) is a null sequence with only nonzero terms. Calculate $\lim_{n\to\infty} \frac{\sin(7\theta_n)}{5\theta_n}$.

Since (θ_n) is a null sequence, $(7\theta_n)$ is also a null sequence. So,

$$\lim_{n\to\infty} \frac{\sin(7\theta_n)}{7\theta_n} = 1.$$

Hence,

$$\lim_{n\to\infty} \frac{\sin(7\theta_n)}{5\theta_n} = \lim_{n\to\infty} \frac{7}{5} \cdot \frac{\sin(7\theta_n)}{7\theta_n}$$ Multiply by $\frac{5}{5}$ and rearrange

$$= \lim_{n\to\infty} \frac{7}{5} \cdot \lim_{n\to\infty} \frac{\sin(7\theta_n)}{7\theta_n}$$ Product limit law

$$= \frac{7}{5} \cdot 1$$ Evaluate limits

$$= \frac{7}{5}.$$

2. Suppose that (θ_n) is a null sequence with only nonzero terms. Calculate $\lim_{n\to\infty} \frac{5-5\cos(2\theta_n)}{\theta_n}$.

Since (θ_n) is a null sequence, $(2\theta_n)$ is also a null sequence. So,

$$\lim_{n\to\infty} \frac{1-\cos(2\theta_n)}{2\theta_n} = 0.$$

Hence,

$$\lim_{n\to\infty} \frac{5-5\cos(2\theta_n)}{\theta_n} = \lim_{n\to\infty} 5 \cdot \frac{(1-\cos(2\theta_n))}{\theta_n}$$ Factor out 5.

$$= \lim_{n\to\infty} \frac{5}{2} \cdot \left(\frac{1-\cos(2\theta_n)}{2\theta_n}\right)$$ Mutiply and Divide by 2

$$= \lim_{n\to\infty} \frac{5}{2} \cdot \lim_{n\to\infty} \frac{1-\cos(2\theta_n)}{2\theta_n}$$ Product limit law

$$= \frac{5}{2} \cdot 0$$ Evaluate limits

$$= 0.$$

3. Suppose that (θ_n) is a null sequence with only nonzero terms. Calculate $\lim\limits_{n\to\infty} \frac{5-5\cos(2\theta_n)}{\theta_n^2}$.

Rewrite like this:

$$\frac{5-5\cos(2\theta_n)}{(\theta_n)^2} = 5 \cdot \frac{1-\cos(2\theta n)}{(\theta n)^2} \cdot \frac{1+\cos(2\theta n)}{1+\cos(2\theta n)}$$

Multiply and divide by $1+\cos(2\theta n)$

$$= \frac{5}{1+\cos(2\theta n)} \cdot \frac{(1-\cos(2\theta n))(1+\cos(2\theta n))}{(\theta n)^2}$$

Rearrange

$$= \frac{5}{1+\cos(2\theta n)} \cdot \frac{\sin^2(2\theta n)}{(\theta n)^2}$$

Multiply numerator and use $1-\cos^2(2\theta n) = \sin^2(2\theta n)$

$$= \frac{5}{1+\cos(2\theta n)} \cdot \frac{\sin(2\theta n)}{\theta n} \cdot \frac{\sin(2\theta n)}{\theta n}$$

Rearrange

$$= \frac{5 \cdot 4}{1+\cos(2\theta n)} \cdot \frac{\sin(2\theta n)}{2\theta n} \cdot \frac{\sin(2\theta n)}{2\theta n}$$

Multiply and divide by 4 and split 4 as $2 \cdot 2$ for the denominator.

Hence,

$$\lim_{n\to\infty} \frac{5-5\cos(2\theta n)}{(\theta n)^2} = \lim_{n\to\infty} \frac{20}{1+\cos(2\theta n)} \cdot \lim_{n\to\infty} \frac{\sin(2\theta n)}{2\theta n} \cdot \lim_{n\to\infty} \frac{\sin(2\theta n)}{2\theta n} = \frac{20}{1+1} \cdot 1 \cdot 1 = 10.$$

4. Suppose that (θ_n) is a null sequence with only nonzero terms. Calculate $\lim\limits_{n\to\infty} \frac{\tan(5\theta_n)}{\theta_n}$.

Rewrite like this:

$$\frac{\tan(5\theta n)}{\theta n} = \frac{\sin(5\theta n)}{\cos(5\theta n)} \cdot \frac{1}{\theta n}$$

Rewrite $\tan(5\theta)$ as $\frac{\sin(5\theta n)}{\cos(5\theta n)}$

$$= \frac{\sin(5\theta n)}{\theta n} \cdot \frac{1}{\cos(\theta n)}$$

Rearrange

$$= \frac{\sin(5\theta n)}{5\theta n} \cdot \frac{5}{\cos(\theta n)}.$$

Multiply and divide by 5.

Hence,

$$\lim_{n\to\infty} \frac{\tan(5\theta n)}{\theta n} = \lim_{n\to\infty} \frac{\sin(5\theta n)}{5\theta n} \cdot \lim_{n\to\infty} \frac{5}{\cos(\theta n)}$$

$$= 1 \cdot \frac{5}{1}$$

$$= 5.$$