1. Write the first four terms of (a_n), where for each natural number n, a_n is given by $a_n = 5^n$.

$a_1 = 5^1 = 5,$

$a_2 = 5^2 = 25,$

$a_3 = 5^3 = 125,$

$a_4 = 5^4 = 625$

2. Show that the sequence (a_n) is increasing and the sequence (b_n) is decreasing, where

a) $a_n = \frac{n+3}{n+5}$

b) $b_n = \frac{n}{n^2+1}$

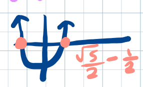
$$a_{n+1} - a_n = \frac{n+1+3}{n+1+5} - \frac{n+3}{n+5}$$
$$= \frac{n+4}{n+6} - \frac{n+3}{n+5}$$
$$= \frac{(n+5)(n+4)}{(n+5)(n+6)} - \frac{(n+6)(n+3)}{(n+5)(n+6)}$$
$$= \frac{n^2+9n+20}{(n+5)(n+6)} - \frac{n^2+9n+18}{(n+5)(n+6)}$$
$$= \frac{2}{(n+5)(n+6)}.$$

Both numerator and denominator are positive for all n, so the entire expression is always positive. Thus $a_{n+1} - a_n > 0$, so a_n is increasing.

$$b_{n+1} - b_n = \frac{n+1}{(n+1)^2+1} - \frac{n}{n^2+1}$$
$$= \frac{n+1}{n^2+2n+2} - \frac{n}{n^2+1}$$
$$= \frac{(n+1)(n^2+1) - n(n^2+2n+2)}{(n^2+2n+2)(n^2+1)}$$
$$= -\frac{n^2+n-1}{(n^2+2n+2)(n^2+1)}$$

rewrite

$$= -\frac{(n+\frac{1}{2})^2 - \frac{5}{4}}{(n^2+2n+2)(n^2+1)}$$



For all n, both numerator and denominator are positive, so the entire expression is negative due to negative sign. Thus $b_{n+1} - b_n < 0$, or $b_{n+1} < b_n$ so b_n is decreasing.

3. Identify an example of a sequence (a_n) with the following properties:

a) (a_n) is strictly increasing and converges to 2

b) (a_n) is strictly decreasing and converges to $\frac{1}{4}$

a) $a_n = 2 - \frac{1}{n}$

b) $a_n = \frac{1}{4} + \frac{1}{n}$

4. Use only the Archimedean property of $\mathbb{R}$ to show that for any positive real number ε, there is a natural number N so that if n is greater than N, then

$$|a_n - L| < \varepsilon,$$

for the following choices of (a_n) and L:

a) $a_n = \frac{9n+2}{3n+4}$ and $L = 3$ b) $a_n = \sqrt{16 + \frac{1}{n}}$ and $L = 4$.

a) Note that for any n,

$$a_n - L = \frac{9n+2}{3n+4} - 3$$
$$= \frac{9n+2-3(3n+4)}{3n+4}$$
$$= -\frac{10}{3n+4},$$

So

$$|a_n - L| = \frac{10}{3n+4}.$$

If $|a_n - L| < \varepsilon$, then

$$|a_n - L| < \varepsilon$$
$$\frac{10}{3n+4} < \varepsilon$$
$$\frac{10}{\varepsilon} < 3n+4$$
$$\frac{10}{\varepsilon} - 4 < 3n$$
$$\underbrace{\frac{10}{3\varepsilon} - \frac{4}{3}}_{x} < n$$

By the Archimedean property, there is a natural number N so that

$$N > \underbrace{\frac{10}{3\varepsilon} - \frac{4}{3}}_{x}$$

So for any $n > N$,

$n > \frac{10}{3\varepsilon} - \frac{4}{3}$ implies that $\frac{10}{3n+4} < \varepsilon$.

Thus for $n \geq N$,
$|a_n - L| < \varepsilon$.

b) Note that for any n,

$$a_n - L = \sqrt{16+\frac{1}{n}} - 4$$
$$= \left(\sqrt{16+\frac{1}{n}} - 4\right)\left(\frac{\sqrt{16+\frac{1}{n}}+4}{\sqrt{16+\frac{1}{n}}+4}\right)$$
$$= \frac{16+\frac{1}{n}-16}{\sqrt{16+\frac{1}{n}}+4}$$
$$= \frac{\frac{1}{n}}{\sqrt{16+\frac{1}{n}}+4}$$
$$= \frac{1}{n}\cdot\frac{1}{\sqrt{16+\frac{1}{n}}+4}$$

So

$$|a_n - L| = \frac{1}{n}\cdot\frac{1}{\sqrt{16+\frac{1}{n}}+4}$$
$$< \frac{1}{n}\cdot\frac{1}{4}.$$

since $\sqrt{16+\frac{1}{n}}+4 > \sqrt{16}+4 = 4$
so $\frac{1}{4} > \frac{1}{\sqrt{16+\frac{1}{n}}+4}$

If $\frac{1}{4n} < \varepsilon$, then $\underbrace{\frac{1}{4\varepsilon}}_{x} < n$.

By the Archimedean property, there is a natural number N so that

$$N > \frac{1}{4\varepsilon}.$$

So for any $n > N$,

$n > \frac{1}{4\varepsilon}$ implies that $\frac{1}{4n} < \varepsilon$.

Thus for $n \geq N$,

$$|a_n - L| < \varepsilon.$$

5. Calculate $\lim_{n\to\infty} \sqrt{100+\frac{1}{n}}$. Carefully justify your reasoning.

For any natural number, $\sqrt{10+\frac{1}{n}} > 10$.
Therefore, there is a positive real number so that

$$\sqrt{100+\frac{1}{n}} = 10+\varepsilon_n$$

Notice that

$$\begin{aligned}
\sqrt{100+\tfrac{1}{n}} &= 10+\varepsilon_n \\
100+\tfrac{1}{n} &= (10+\varepsilon_n)^2 \\
100+\tfrac{1}{n} &= 100+20\varepsilon_n+\varepsilon_n^2 \\
\tfrac{1}{n} &= 20\varepsilon_n+\varepsilon_n^2
\end{aligned}$$

Since ε_n is positive,

$$\frac{1}{n} = 20\varepsilon_n+\varepsilon_n^2 > \varepsilon_n^2$$

implies that

$$\varepsilon_n < \frac{1}{\sqrt{n}}.$$

Since

$$0 < \varepsilon_n < \frac{1}{\sqrt{n}},$$

(ε_n) is bounded above and below by null sequences. Therefore by the Squeeze theorem for null sequence, (ε_n) is a null sequence.

Thus, by the sum limit law,

$$\begin{aligned}
\lim_{n\to\infty}\sqrt{100+\tfrac{1}{n}} &= \lim_{n\to\infty}(10+\varepsilon_n) \\
&= 10+0 \\
&= 10.
\end{aligned}$$

6. Take (a_n), (b_n), and (c_n) to be sequences with

$$\lim_{n\to\infty} a_n = 5, \quad \lim_{n\to\infty} b_n = 1, \quad \text{and} \quad \lim_{n\to\infty} c_n = -2.$$

Use the limit laws to compute the following:

$$\lim_{n\to\infty} \frac{(a_n)^3+2b_n}{c_n+\frac{1}{n^5}}.$$

Use the limit laws to obtain that

$$\lim_{n\to\infty}\frac{(a_n)^3+2b_n}{c_n+\frac{1}{n^5}} = \frac{\lim_{n\to\infty}\left((a_n)^3+2b_n\right)}{\lim_{n\to\infty}\left(c_n+\frac{1}{n^5}\right)} \qquad \text{quotient limit law}$$

$$= \frac{\lim_{n\to\infty}(a_n)^3+\lim_{n\to\infty}2b_n}{\lim_{n\to\infty}c_n+\lim_{n\to\infty}\frac{1}{n^5}} \qquad \text{sum limit law}$$

$$= \frac{\lim_{n\to\infty}a_n\lim_{n\to\infty}a_n\lim_{n\to\infty}a_n+\lim_{n\to\infty}2\lim_{n\to\infty}b_n}{\lim_{n\to\infty}c_n+\lim_{n\to\infty}\frac{1}{n^5}} \qquad \text{product limit law}$$

$$= \frac{5\cdot5\cdot5+2\cdot1}{-2+0} \qquad \text{evaluate limits}$$

$$= -\frac{127}{2}$$

7. Use the limit laws to determine the following limits:

a) $\lim\limits_{n\to\infty} \frac{5n}{3n-1}$ b) $\lim\limits_{n\to\infty} \frac{n^2+4n}{n^5+n^2+2}$

a) Rewrite like this

$$\lim_{n\to\infty} \frac{5n}{3n-1} = \lim_{n\to\infty} \frac{5n}{3n-1} \cdot \frac{\frac{1}{n}}{\frac{1}{n}}$$ multiply numerator and denominator

$$= \lim_{n\to\infty} \frac{5}{3-\frac{1}{n}}$$

$$= \frac{\lim_{n\to\infty} 5}{\lim_{n\to\infty} 3 - \lim_{n\to\infty} \frac{1}{n}}$$ quotient and sum limit law

$$= \frac{5}{3-0}$$ evaluate limit

$$= \frac{5}{3}$$

b) Rewrite like this

$$\lim_{n\to\infty} \frac{n^2+4n}{n^5+n^2+2} = \lim_{n\to\infty} \frac{n^2+4n}{n^5+n^2+2} \cdot \frac{\frac{1}{n^5}}{\frac{1}{n^5}}$$ multiply both numerator and denominator

$$= \lim_{n\to\infty} \frac{\frac{1}{n^3}+\frac{4}{n^4}}{1+\frac{1}{n^3}+\frac{2}{n^5}}$$

$$= \frac{\lim_{n\to\infty} \frac{1}{n^3} + \lim_{n\to\infty} \frac{4}{n^4}}{\lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{1}{n^3} + \lim_{n\to\infty} \frac{2}{n^3}}$$ quotient and sum law

$$= \frac{0+0}{1+0+0}$$ evaluate limit

$$= \frac{0}{1}$$

$$= 0$$

8. Take (a_n) and (b_n) to be sequences with

$$\lim_{n\to\infty} a_n = 2 \quad \text{and} \quad \lim_{n\to\infty} \frac{b_n}{a_n} = 18.$$

Carefully justify that (b_n) is convergent and calculate its limit.

Since (a_n) converges to a non-zero limit, there exists an N in $\mathbb{N}$ so that $a_n \neq 0$ for all $n \geq N$.

Rewrite b_n like this

$$b_n = b_n \cdot \frac{a_n}{a_n}$$

$$= \frac{b_n}{a_n} \cdot a_n,$$

for $n \geq N$.

So

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{b_n}{a_n} \lim_{n\to\infty} a_n$$

$$= 18 \cdot 2$$ evaluate limit

$$= 36$$

9. Calculate $\lim_{n\to\infty} n\left(\sqrt{100+\frac{1}{n}}-10\right)$. Carefully justify your reasoning.

Rewrite like this:

$$\lim_{n\to\infty} n\left(\sqrt{100+\tfrac{1}{n}}-10 \cdot \frac{\sqrt{100+\frac{1}{n}}+10}{\sqrt{100+\frac{1}{n}}+10}\right) = \lim_{n\to\infty} n \frac{100+\frac{1}{n}-100}{\sqrt{100+\frac{1}{n}}+10}$$ multiply numerator

$$= \lim_{n\to\infty} n \frac{\frac{1}{n}}{\sqrt{100+\frac{1}{n}}+10}$$

$$= \lim_{n\to\infty} n \cdot \frac{1}{n} \cdot \frac{1}{\sqrt{100+\frac{1}{n}}+10}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{100+\frac{1}{n}}+10}$$

$$= \frac{1}{10+10}$$ evaluate limit

$$= \frac{1}{20}$$

10. Take a_n to be the sequence that is given by

$$a_n = \frac{20n^2+5n\cos(n)}{4n^2+4}.$$

Calculate $\lim_{n\to\infty} a_n$. Carefully justify your reasoning.

① upper bound

For all n, $\cos(n) \leq 1$.

Since $20n^2$ and $5n$ are always positive, we have

$$20n^2+5n\cos(n) \leq 20n^2+5n.$$

Since $4n^2+4$ is always positive,

$$\underbrace{\frac{20n^2+5n\cos(n)}{4n^2+4}}_{a_n} \leq \underbrace{\frac{20n^2+5n}{4n^2+4}}_{b_n}$$

② lower bound

For all n, $-1 \leq \cos(n)$.

we have

$$20n^2+5n\cos(n) \geq 20n^2-5n$$

Since $4n^2+4$ is always positive,

$$\underbrace{\frac{20n^2+5n\cos(n)}{4n^2+4}}_{a_n} \geq \underbrace{\frac{20n^2-5n}{4n^2+4}}_{c_n}$$

③ conclusion

Since

$$c_n \leq a_n \leq b_n$$

and

$$\lim_{n\to\infty} c_n = 5, \quad \lim_{n\to\infty} b_n = 5,$$

we have by the squeeze theorem that

$$\lim_{n\to\infty} \frac{20n^2+5n\cos(n)}{4n^2+4} = 5.$$

11. Determine whether the sequence diverges to either infinity or negative infinity:

a) $a_n = \left(\frac{1}{8}\right)^{-n}$ b) $a_n = \ln(n+1)$ c) $a_n = -n^5$

$$\lim_{n\to\infty} \left(\frac{1}{8}\right)^{-n} = \infty$$

diverges to infinity

$$\lim_{n\to\infty} \ln(n+1) = \infty$$

diverges to infinity

$$\lim_{n\to\infty} -n^5 = -\infty$$

diverges to negative infinity